Fourier Transform on $\mathbb{R}$: Motivation, Properties, and Inversion

lecture notes for the c5 presentation

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Abstract

These notes provide a self-contained companion to a 50-minute talk on the Fourier transform on $\mathbb{R}$ (Stein–Shakarchi normalization). We develop the transform from translation symmetry, prove the core identities, construct Gaussian approximate identities, and give a detailed proof of Fourier inversion for $f \in L^1(\mathbb{R})$ with $\widehat{f} \in L^1(\mathbb{R})$. Worked examples (Gaussian, box→sinc, triangle via convolution) and practical remarks are included. Short statements of Plancherel/Parseval and Hausdorff–Young appear as further directions.

1 Motivation and normalization

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Hook: *We want to understand functions by their oscillations.* The real line has a symmetry—translation. The functions that "diagonalize" translation are complex exponentials. Projecting a function onto these gives the Fourier transform.

On $\mathbb{R}$, define the translation operator $T_t f(x) := f(x - t)$. The family $\{T_t\}_{t\in\mathbb{R}}$ forms a group: $T_s T_t = T_{s+t}$, $T_0 = \mathrm{Id}$, and $T_{-t} = T_t^{-1}$.

We seek eigenfunctions ϕ of all translations:

$$T_t\phi = \lambda(t)\phi \quad \forall t \iff \phi(x-t) = \lambda(t)\phi(x).$$

Fix x and define $\lambda(t) = \phi(x-t)/\phi(x)$ (independent of x). Then $\lambda(t+s) = \lambda(t)\lambda(s)$ and $\lambda(0) = 1$. Under mild regularity (continuity/measurability), $\lambda(t) = \mathrm{e}^{ct}$ for some $c \in \mathbb{C}$. To make translations unitary on L^2, we require $|\lambda(t)| = 1$, hence $c = 2\pi\mathrm{i}\xi$ with $\xi \in \mathbb{R}$ and eigenfunctions

$$\chi_\xi(x) = \mathrm{e}^{2\pi\mathrm{i}\xi x}.$$

Definition 1.1 (Fourier transform (Stein–Shakarchi normalization)). For $f \in L^1(\mathbb{R})$, the Fourier transform is

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,dx, \qquad \xi \in \mathbb{R}.$$

Remark 1.2. Different texts place the 2π differently. We stick with Stein–Shakarchi's convention so that the Gaussian $g(x) = \mathrm{e}^{-\pi x^2}$ is self-dual: $\widehat{g} = g$.

Takeaway

The kernel $\mathrm{e}^{-2\pi\mathrm{i}x\xi}$ is *forced by translation symmetry*: complex exponentials are the characters of $(\mathbb{R}, +)$.

2 Preliminaries: spaces, tools, and lemmas

We use the usual $L^p(\mathbb{R})$ spaces with norms $\|f\|_p = (\int |f|^p)^{1/p}$ for $1 \le p < \infty$, $\|f\|_\infty = \operatorname{ess\,sup}|f|$. The Schwartz class $\mathcal{S}(\mathbb{R})$ consists of smooth functions with rapid decay of all derivatives; $\mathcal{S}$ is dense in L^p for $1 \le p < \infty$.

Definition 2.1 (Convolution). For $f, g \in L^1(\mathbb{R})$,

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)\,g(y)\,dy.$$

Then $f * g \in L^1(\mathbb{R})$ and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$.

Lemma 2.2 (Tonelli/Fubini, Dominated Convergence; summaries). *If $\iint |h(x,y)|\,dx\,dy < \infty$, then $\int(\int h\,dx)\,dy = \int(\int h\,dy)\,dx$. If $h_n \to h$ pointwise and $|h_n| \le g$ with $g \in L^1$, then $\int h_n \to \int h$.*

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Call out when you're using these: convolution theorem (Fubini), damping/inversion (DCT), Lebesgue differentiation (pointwise recovery).

3 First properties of the Fourier transform

Proposition 3.1 (Basic bounds and continuity). *If $f \in L^1(\mathbb{R})$ then $\|\widehat{f}\|_\infty \le \|f\|_1$ and $\widehat{f}$ is uniformly continuous.*

Proof. Triangle inequality gives $|\widehat{f}(\xi)| \le \int |f(x)|dx = \|f\|_1$. Uniform continuity follows from $|\mathrm{e}^{-2\pi\mathrm{i}x\xi} - \mathrm{e}^{-2\pi\mathrm{i}x\eta}| \le 2\pi|x|\,|\xi - \eta|$ together with approximation by $f \in L^1 \cap L^1(|x|dx)$ or a standard cut-off/density argument. $\square$

Proposition 3.2 (Linearity, conjugation, reflection). *For $a, b \in \mathbb{C}$, $\widehat{(af + bg)} = a\widehat{f} + b\widehat{g}$. If $\overline{f}$ is the complex conjugate, then $\widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)}$. If $f^\sharp(x) := f(-x)$ then $\widehat{f^\sharp}(\xi) = \widehat{f}(-\xi)$.*

Proposition 3.3 (Translation/modulation, scaling, differentiation/multiplication). *Let $t \in \mathbb{R}$, $a \neq 0$.*

$$\widehat{(T_t f)}(\xi) = \mathrm{e}^{-2\pi \mathrm{i} t\xi}\, \widehat{f}(\xi),$$
$$(\widehat{\mathrm{e}^{2\pi \mathrm{i}\xi_0 x} f})(\xi) = \widehat{f}(\xi - \xi_0),$$
$$(\widehat{f(ax)})(\xi) = \frac{1}{|a|}\, \widehat{f}\left(\frac{\xi}{a}\right).$$

If $f, f' \in L^1$, then $\widehat{(f')}(\xi) = (2\pi \mathrm{i}\xi)\widehat{f}(\xi)$, and if $xf(x) \in L^1$ then $\widehat{(xf)}(\xi) = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{f}(\xi)$.

Proposition 3.4 (Convolution theorem). *If $f, g \in L^1(\mathbb{R})$, then $\widehat{(f * g)} = \widehat{f} \cdot \widehat{g}$.*

Proof. By Tonelli,

$$\widehat{(f * g)}(\xi) = \iint f(x-y)g(y)\, \mathrm{e}^{-2\pi \mathrm{i}x\xi}\, dy\, dx = \int g(y)\, \mathrm{e}^{-2\pi \mathrm{i}y\xi}\, dy \cdot \int f(u)\, \mathrm{e}^{-2\pi \mathrm{i}u\xi}\, du.$$

□

Lemma 3.5 (Riemann–Lebesgue). *If $f \in L^1(\mathbb{R})$ then $\widehat{f}$ is bounded/UC and $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

Proof sketch. Approximate f in L^1 by C_c^1 functions; integrate by parts on each compact interval to get a $1/|\xi|$ bound, then pass to the limit. □

Takeaway

Time–domain actions become simple frequency rules: translation↔phase, convolution↔multiplication, differentiation↔polynomial factor in ξ.

4 Approximate identities and Gaussians

Let $g(x) = \mathrm{e}^{-\pi x^2}$ (so $\widehat{g} = g$). For $\varepsilon > 0$ define the (positive) approximate identity

$$\phi_\varepsilon(x) := \frac{1}{\varepsilon}\, g\left(\frac{x}{\varepsilon}\right) = \frac{1}{\varepsilon}\, \mathrm{e}^{-\pi(x/\varepsilon)^2}, \qquad \int \phi_\varepsilon = 1.$$

By scaling,

$$\widehat{\phi_\varepsilon}(\xi) = \widehat{g}(\varepsilon\xi) = \mathrm{e}^{-\pi\varepsilon^2\xi^2}.$$

Proposition 4.1 (Approximate identity). *For $f \in L^1(\mathbb{R})$, $f * \phi_\varepsilon \to f$ in L^1 as $\varepsilon \downarrow 0$, and at every Lebesgue point x of f, $(f * \phi_\varepsilon)(x) \to f(x)$.*

Remark 4.2 (Lebesgue points). By the Lebesgue Differentiation Theorem, almost every $x \in \mathbb{R}$ is a Lebesgue point of $f \in L^1$.

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Say: "ϕ_ε is a bell that concentrates at 0, has mass 1. Convolving with it gently averages f near x; as $\varepsilon \to 0$ the average becomes a point evaluation at Lebesgue points."

5 Fourier inversion

5.1 Statement and overall plan

Theorem 5.1 (Fourier inversion, L^1 version)**.** *If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi \quad \text{for almost every } x \in \mathbb{R}.$$

Equivalently, $\mathcal{F}^{-1}(\widehat{f}) = f$ a.e.

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Roadmap: (1) prove for Schwartz f using Gaussians; (2) extend to L^1 using dominated convergence on the frequency side and the approximate-identity limit on the time side at Lebesgue points.

5.2 Schwartz case via Gaussian damping

Lemma 5.2 (Key identity)**.** *For $f \in \mathcal{S}(\mathbb{R})$ and $\varepsilon > 0$,*

$$(f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi.$$

Proof. By the convolution theorem, $\widehat{(f * \phi_\varepsilon)} = \widehat{f} \cdot \widehat{\phi_\varepsilon} = \widehat{f}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}$. Apply inverse transform inside $\mathcal{S}$. ☐

Proposition 5.3 (Inversion for Schwartz)**.** *If $f \in \mathcal{S}(\mathbb{R})$, then for every $x \in \mathbb{R}$,*

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi.$$

Proof. For each x, Lemma 5.2 gives $(f * \phi_\varepsilon)(x) = \int \widehat{f}(\xi)\mathrm{e}^{2\pi \mathrm{i} x\xi}\mathrm{e}^{-\pi\varepsilon^2\xi^2} d\xi$. Since $\widehat{f} \in \mathcal{S} \subset L^1$ and $|\mathrm{e}^{-\pi\varepsilon^2\xi^2}| \le 1$, by DCT,

$$\lim_{\varepsilon \downarrow 0} \int \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi = \int \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi} d\xi.$$

But $f * \phi_\varepsilon \to f$ uniformly (even in the Schwartz topology), so the left-hand side tends to $f(x)$. ☐

5.3 Extension to L^1 with $\widehat{f} \in L^1$

Proof of Theorem 5.1. Let $f \in L^1$ with $\widehat{f} \in L^1$. For $\varepsilon > 0$, the identity of Lemma 5.2 holds for approximants $f_n \in \mathcal{S}$ with $f_n \to f$ in L^1 and $\widehat{f_n} \to \widehat{f}$ in L^1 (standard density + continuity of $\mathcal{F}$ on $\mathcal{S}$). Passing to the limit gives, for almost every x,

$$(f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi.$$

Since $|\widehat{f}(\xi)\mathrm{e}^{-\pi\varepsilon^2\xi^2}| \le |\widehat{f}(\xi)| \in L^1$, DCT yields

$$\lim_{\varepsilon \downarrow 0} \int \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi = \int \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi.$$

At the same time, $f * \phi_\varepsilon \to f$ at every Lebesgue point x of f (a.e. x). Combining the limits gives the claim. ☐

Remark 5.4 (Continuity points and jump points). If $f \in L^1$ is continuous at x, then the inversion integral converges to $f(x)$. At a jump discontinuity, reasonable summation methods (e.g. symmetric limits) return the midpoint value. This mirrors the Fourier series setting (Gibbs phenomenon).

Takeaway

Inversion is proved by *regularize, invert, and de-regularize.* Gaussian damping controls the integral; letting $\varepsilon \downarrow 0$ removes the damping and recovers f almost everywhere.

6 Worked examples

Example 6.1 (Gaussian is self-dual). Let $g(x) = \mathrm{e}^{-\pi x^2}$. Completing the square,

$$\widehat{g}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\pi x^2} \, \mathrm{e}^{-2\pi \mathrm{i} x\xi} \, dx = \mathrm{e}^{-\pi\xi^2} \int_{\mathbb{R}} \mathrm{e}^{-\pi(x-\mathrm{i}\xi)^2} dx = \mathrm{e}^{-\pi\xi^2}.$$

(The contour shift is justified for Schwartz functions, or by recognizing the standard Gaussian integral.)

Example 6.2 (Box $\to$ sinc). For $f = \mathbf{1}_{[-1,1]}$,

$$\widehat{f}(\xi) = \int_{-1}^{1} \mathrm{e}^{-2\pi \mathrm{i} x\xi} dx = \frac{\sin(2\pi\xi)}{\pi\xi}.$$

The $\sim 1/|\xi|$ decay reflects the jump discontinuities.

Example 6.3 (Triangle via convolution). Let $\tau = f * f$ with $f = \mathbf{1}_{[-1,1]}$. Then $\widehat{\tau} = \widehat{f}^2 = \left(\frac{\sin(2\pi\xi)}{\pi\xi}\right)^2$. The extra convolution smooths τ and squares the sinc, giving $\sim 1/\xi^2$ decay.

7 Further directions (statements only)

7.1 Plancherel/Parseval on L^2

Theorem 7.1 (Plancherel). *$\mathcal{F}$ extends uniquely to a unitary operator on $L^2(\mathbb{R})$:*

$$\|f\|_2 = \|\widehat{f}\|_2, \qquad \langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle.$$

In particular, L^2-inversion holds with no L^1 assumption on $\widehat{f}$.

Remark 7.2 (Proof idea). Prove on $\mathcal{S}$ (Gaussian integral and Fubini), use density of $\mathcal{S}$ in L^2, and extend by continuity; unitarity gives inversion.

7.2 Hausdorff–Young inequality

Theorem 7.3 (Hausdorff–Young). *If $1 \le p \le 2$ and $1/p + 1/q = 1$, then for $f \in L^p(\mathbb{R})$ we have $\widehat{f} \in L^q(\mathbb{R})$ and $\|\widehat{f}\|_q \le \|f\|_p$ (with the Stein–Shakarchi normalization).*

Remark 7.4 (Sketch). Interpolate between the endpoints: $\mathcal{F} : L^1 \to L^\infty$ (trivial bound) and $\mathcal{F} : L^2 \to L^2$ (Plancherel) using Riesz–Thorin.

References and further reading

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