A CENTRAL LIMIT THEOREM FOR RANDOM EULER-PRODUCT SURROGATES OF $\log|\zeta(1/2+it)|$

IZ TECÚM

Abstract. We study a classical random Euler product model for the size of $\zeta(1/2+it)$ by assigning i.i.d. random phases $(\theta_p)_{p\le y}$ and considering the random Euler product $Z_y(t) = \prod_{p\le y}\big(1-e^{i\theta_p}p^{-1/2-it}\big)^{-1}$. Writing $X_y(t) = \mathrm{Re}\log Z_y(t)$, we prove that after normalization by $\sigma_y^2 = \frac{1}{2}\sum_{p\le y}\frac{1}{p}$ one has a central limit theorem $X_y(t)/\sigma_y \Rightarrow N(0,1)$ as $y\to\infty$ (in particular the law is independent of t). The argument splits $X_y(t) = S_y(t) + R_y(t)$ into the main layer $S_y(t) = \sum_{p\le y}\cos(\theta_p - t\log p)/\sqrt{p}$, handled via Lindeberg–Feller, and a higher-power remainder $R_y(t) = \sum_{p\le y}\sum_{k\ge 2}\cos(k(\theta_p - t\log p))/(k\,p^{k/2})$, which is uniformly $O_{L^2}(1)$ and hence $o_{\mathbb{P}}(\sigma_y)$ since $\sigma_y\to\infty$. We record a companion numerical experiment illustrating Gaussianity of $S_y(t)/\sigma_y$ and the Mertens-scale growth $\sigma_y^2 \sim \frac{1}{2}\log\log y$.

Contents

Date: January 18, 2026.

1. Introduction

The size of $\zeta(1/2+it)$ is governed, at least formally, by its Euler product and the resulting prime-by-prime oscillations. On the critical line one does not have absolute convergence, but the standard heuristic is to treat prime phases $p^{-it} = e^{-it\log p}$ as if they behaved like "independent rotations" and to study the logarithm of a truncated Euler product. This point of view goes back to classical probabilistic models for $\log\zeta(1/2+it)$ and predicts (i) a Gaussian fluctuation scale and (ii) the Mertens-type variance growth $\asymp \log\log$ of the cutoff.

In this note we work with a clean random surrogate in which the arithmetic oscillations are kept while the multiplicative structure is randomized at the primes. Fix $y \geq 2$ and take i.i.d. phases $\theta_p \sim \mathrm{Unif}[0, 2\pi)$ for $p \leq y$. For $t \in \mathbb{R}$ define the random Euler product

$$Z_y(t) = \prod_{p\leq y}\left(1 - \frac{e^{i\theta_p}}{p^{1/2+it}}\right)^{-1}, \qquad X_y(t) := \operatorname{Re}\log Z_y(t),$$

where log is understood through its absolutely convergent series expansion. Our main result is a central limit theorem for $X_y(t)$ after the natural normalization

$$\sigma_y^2 := \frac{1}{2}\sum_{p\leq y}\frac{1}{p}, \qquad \text{so that} \qquad \sigma_y^2 \sim \tfrac{1}{2}\log\log y \quad (y\to\infty),$$

and in particular $X_y(t)/\sigma_y$ converges in distribution to a standard normal.

The proof is intentionally elementary and isolates the underlying mechanism. Expanding the logarithm gives a trigonometric series indexed by primes and Euler-product powers; separating the $k=1$ layer from $k \geq 2$ yields the decomposition $X_y(t) = S_y(t) + R_y(t)$ with

$$S_y(t) = \sum_{p\leq y}\frac{\cos(\theta_p - t\log p)}{\sqrt{p}}, \qquad R_y(t) = \sum_{p\leq y}\sum_{k\geq 2}\frac{\cos(k(\theta_p - t\log p))}{k\,p^{k/2}}.$$

The main term $S_y(t)$ is a sum of independent, uniformly bounded, mean-zero variables whose total variance is $\sigma_y^2 \to \infty$, so Lindeberg–Feller gives $S_y(t)/\sigma_y \Rightarrow N(0,1)$. The remainder $R_y(t)$ is square-summable in p (because $p^{-k/2}$ is rapidly decaying for $k \geq 2$), hence $R_y(t) = O_{L^2}(1)$ uniformly in y and therefore $R_y(t) = o_{\mathbb{P}}(\sigma_y)$. Slutsky's theorem transfers the CLT from $S_y(t)$ to $X_y(t)$. A basic shift symmetry of the uniform phases also implies that the law of $X_y(t)$ does not depend on t, though we retain t to match the number-theoretic notation.

2. Model and Euler-product expansion

2.1. **Random Euler product (primes up to y).** Fix a cutoff $y \geq 2$ and let $(\theta_p)_{p \leq y}$ be i.i.d. random variables, each uniform on $[0, 2\pi)$. It is often convenient to package these phases as *Steinhaus variables*

$$\xi_p := e^{i\theta_p} \qquad (p \leq y),$$

so that ξ_p are i.i.d. on the unit circle with $\mathbb{E}(\xi_p^k) = 0$ for every nonzero integer k (see, e.g., [1, 2, 12]).

For $t \in \mathbb{R}$, define the random Euler product

$$Z_y(t) \ := \ \prod_{p \leq y} \left(1 - \frac{e^{i\theta_p}}{p^{1/2+it}}\right)^{-1} = \prod_{p \leq y} \left(1 - \xi_p \, p^{-1/2-it}\right)^{-1}. \tag{1}$$

This should be read as a finite Euler product surrogate for $\zeta(1/2+it)$, with the arithmetic phases $p^{-it} = e^{-it\log p}$ retained and the multiplicative signs replaced by independent random phases; see standard references for Euler products and their logarithms [1, 2, 10]. (We will only use the prime-sum technology that underlies these expansions; no deep zeta-function results are needed.)

We study the real part of the logarithm,

$$X_y(t) \ := \ \operatorname{Re} \log Z_y(t), \tag{2}$$

where log is interpreted via an absolutely convergent series (proved below), so there is no ambiguity from branch choices; compare the standard use of logarithmic Euler-product expansions in multiplicative number theory [1, 2].

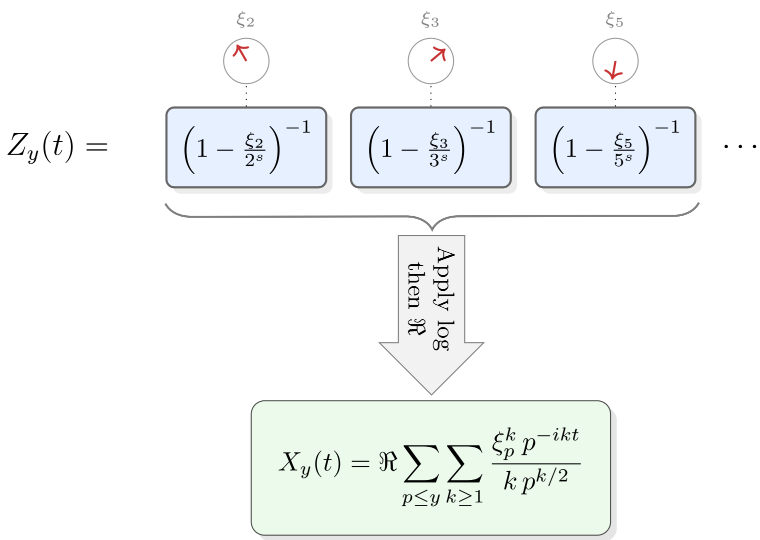

Figure 1. A finite random Euler product and its logarithmic expansion. The randomness is in the phases $\xi_p = e^{i\theta_p}$, while the deterministic oscillation p^{-it} remains. This "log Euler product $\rightsquigarrow$ prime sum" mechanism is standard in multiplicative number theory [1, 2].

2.2. **A basic symmetry: t is inessential in distribution.** The dependence on t is a notational convenience rather than a probabilistic one; compare the same shift-invariance idea in standard probability references [3, 4, 6].

Lemma 2.1 (Uniform shift symmetry). *Let $\Theta \sim \mathrm{Unif}[0, 2\pi)$ and let $\alpha \in \mathbb{R}$ be deterministic. Then $\Theta - \alpha \pmod{2\pi}$ is also uniform on $[0, 2\pi)$.*

Proof. For any interval $I \subset [0, 2\pi)$, the event $\{\Theta - \alpha \in I \pmod{2\pi}\}$ is equivalent to $\{\Theta \in I + \alpha \pmod{2\pi}\}$. Since Lebesgue measure on the circle is translation-invariant,

$$\mathbb{P}(\Theta - \alpha \in I \pmod{2\pi}) = \frac{|I + \alpha|}{2\pi} = \frac{|I|}{2\pi}.$$

Thus $\Theta - \alpha$ modulo 2π is uniform. $\square$

Proposition 2.2 (The law does not depend on t). *For each fixed y, the random variables $X_y(t)$ have the same distribution for all $t \in \mathbb{R}$. Equivalently, one may set $t = 0$ throughout without changing the law.*

Proof. Define for each prime $p \le y$ the shifted phase

$$U_p(t) := \theta_p - t \log p \pmod{2\pi}.$$

By Lemma 2.1, $U_p(t)$ is uniform on $[0, 2\pi)$ for fixed t, and since the θ_p are independent, the family $(U_p(t))_{p \le y}$ is i.i.d. uniform as well. In the expansion (4) below, $X_y(t)$ is a measurable function of the i.i.d. family $(U_p(t))_{p \le y}$, so its law is the same as at $t = 0$. $\square$

Remark 2.3 (How we use this). We keep t to mirror the $\zeta(1/2 + it)$ heuristic, but all probabilistic statements in this paper are uniform in t and can be proved with $t = 0$ for readability; see [10, 1] for the corresponding deterministic number-theory viewpoint where t is the varying parameter.

2.3. **Logarithmic series and the cosine expansion.** Because each Euler factor is strictly inside the unit disk, the logarithm can be expanded absolutely; this is the standard device used throughout the analytic number theory literature on Euler products [1, 2].

Lemma 2.4 (Absolute convergence of the logarithmic expansion). *Fix $y \ge 2$ and $t \in \mathbb{R}$. For each prime $p \le y$,*

$$\left| \frac{e^{i\theta_p}}{p^{1/2+it}} \right| = \frac{1}{\sqrt{p}} < 1,$$

and therefore the series identity

$$-\log(1 - z) = \sum_{k \ge 1} \frac{z^k}{k} \qquad (|z| < 1)$$

applies with $z = \xi_p p^{-1/2-it}$. Consequently, $\log Z_y(t)$ admits the absolutely convergent expansion

$$\log Z_y(t) = \sum_{p \le y} \sum_{k \ge 1} \frac{e^{ik(\theta_p - t \log p)}}{k\, p^{k/2}}. \tag{3}$$

Moreover, the series converges absolutely for each fixed realization of (θ_p), and the order of summation over the finite set of primes and the infinite index k is justified.

Proof. Fix $p \le y$ and set $z = \xi_p p^{-1/2-it}$, so $|z| = p^{-1/2} < 1$. Then

$$\log \left(1 - \xi_p p^{-1/2-it}\right)^{-1} = -\log(1 - z) = \sum_{k \ge 1} \frac{z^k}{k} = \sum_{k \ge 1} \frac{\xi_p^k\, p^{-ikt}}{k\, p^{k/2}}.$$

Summing over the *finite* set of primes $p \le y$ gives (3). Absolute convergence is immediate from

$$\sum_{k \ge 1} \left| \frac{z^k}{k} \right| \le \sum_{k \ge 1} |z|^k = \frac{|z|}{1 - |z|} = \frac{p^{-1/2}}{1 - p^{-1/2}} < \infty.$$

Since the prime sum is finite, exchanging the finite sum and the absolutely convergent k-sum is legitimate. □

Taking real parts yields a completely explicit trigonometric series.

Corollary 2.5 (Cosine expansion for $X_y(t)$)**.** *For each fixed $y \geq 2$ and $t \in \mathbb{R}$,*

$$X_y(t) = \sum_{p \leq y} \sum_{k \geq 1} \frac{\cos\big(k(\theta_p - t \log p)\big)}{k\, p^{k/2}}. \tag{4}$$

Proof. Take real parts term-by-term in (3). This is justified by absolute convergence. □

2.4. **Main term vs remainder decomposition.** The random trigonometric series (4) naturally splits into a first-order term (the "$k = 1$ layer") and a higher-order remainder (all $k \geq 2$). Define

$$S_y(t) \;:=\; \sum_{p \leq y} \frac{\cos(\theta_p - t \log p)}{\sqrt{p}}, \tag{5}$$

and

$$R_y(t) \;:=\; \sum_{p \leq y} \sum_{k \geq 2} \frac{\cos\big(k(\theta_p - t \log p)\big)}{k\, p^{k/2}}, \tag{6}$$

so that

$$X_y(t) = S_y(t) + R_y(t). \tag{7}$$

This "first layer drives the variance; higher layers are square-summable" philosophy is a standard theme in random Euler product models [12, 2], and it is also consistent with the deterministic use of truncations in the zeta-function literature [10].

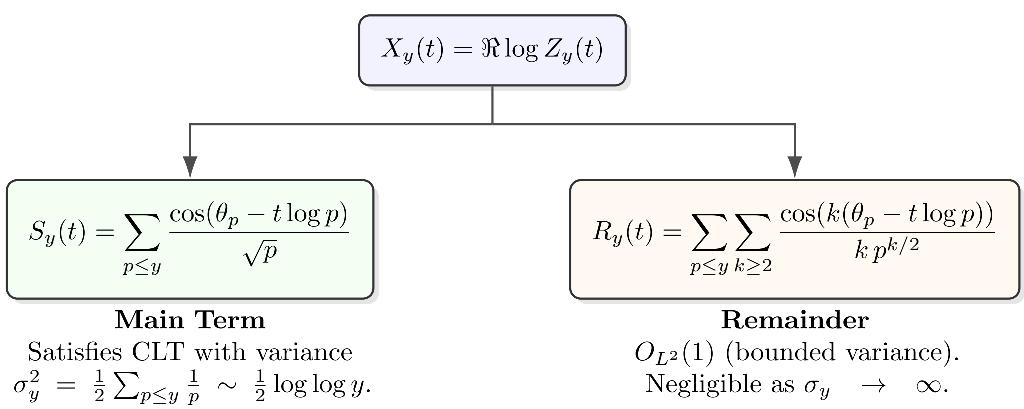

FIGURE 2. The structural split $X_y = S_y + R_y$. The first layer S_y is a sum of independent bounded variables with slowly growing variance; the higher layers R_y are square-summable in p and remain uniformly controlled.

2.5. **Geometric intuition: a random walk of prime vectors.** The main term $S_y(t)$ is the real projection of a complex-valued random walk. Define

$$V_y(t) \ := \sum_{p\le y} \frac{e^{i(\theta_p - t\log p)}}{\sqrt{p}} \qquad \text{so that} \qquad S_y(t) = \mathrm{Re}\ V_y(t).$$

Each prime contributes a vector of length $p^{-1/2}$ with a random direction (uniform phase). The cutoff y controls how many steps we take, with step lengths decaying as $p^{-1/2}$. This picture is the heuristic reason one expects Gaussian behavior after normalization.

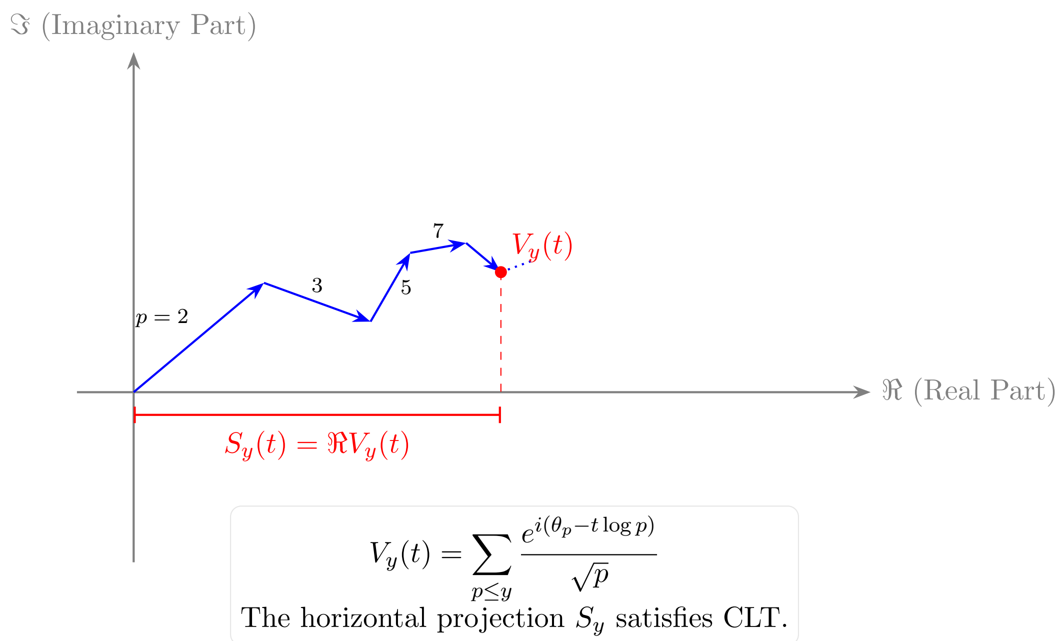

FIGURE 3. A stylized view of the prime-indexed random walk $V_y(t)$. Each step has random direction and length $1/\sqrt{p}$. Gaussian behavior emerges after normalization because the variance accumulates like $\sum_{p\le y} 1/p \sim \log\log y$.

2.6. **Moments of the cosine modes (orthogonality on the circle).** The trigonometric structure in (4) is governed by basic Fourier orthogonality [3, 4, 5].

Lemma 2.6 (Fourier orthogonality for cosine modes)**.** *Let U be uniform on $[0, 2\pi)$. Then for integers $k, \ell \ge 1$,*

$$\mathbb{E}\big[\cos(kU)\big] = 0, \qquad \mathbb{E}\big[\cos(kU)\cos(\ell U)\big] = \begin{cases} \frac{1}{2}, & k = \ell, \\ 0, & k \ne \ell. \end{cases}$$

Proof. The first identity is $\mathbb{E}[\cos(kU)] = (2\pi)^{-1}\int_0^{2\pi} \cos(ku)\,du = 0$ [3, 4]. For the second, use the product-to-sum identity

$$\cos(ku)\cos(\ell u) = \frac{1}{2}\cos((k-\ell)u) + \frac{1}{2}\cos((k+\ell)u)$$

and integrate over $[0, 2\pi]$ [5]. The integral vanishes unless $k = \ell$, in which case the first cosine term is $\cos(0u) = 1$ and contributes $1/2$. □

2.7. **Mean and variance: the natural normalization.** Because the primes contribute independently, the first-order term $S_y(t)$ has a transparent variance [3, 4].

Proposition 2.7 (Centering and variance of the main term)**.** *For every $t \in \mathbb{R}$,*

$$\mathbb{E}[S_y(t)] = 0, \qquad \mathrm{Var}(S_y(t)) = \frac{1}{2}\sum_{p \le y} \frac{1}{p}.$$

In particular, with the variance parameter

$$\sigma_y^2 \;:=\; \mathrm{Var}(S_y(t)) = \frac{1}{2}\sum_{p \le y} \frac{1}{p}, \tag{8}$$

the quantity σ_y^2 is increasing in y and $\sigma_y \to \infty$ as $y \to \infty$.

Proof. Write $U_p(t) = \theta_p - t\log p \pmod{2\pi}$ as in Proposition 2.2. Then $U_p(t)$ are i.i.d. uniform. Hence $\mathbb{E}[\cos(U_p(t))] = 0$ by Lemma 2.6, so $\mathbb{E}[S_y(t)] = 0$.

For the variance, independence across primes gives

$$\mathrm{Var}(S_y(t)) = \sum_{p \le y} \mathrm{Var}\left(\frac{\cos(U_p(t))}{\sqrt{p}}\right) = \sum_{p \le y} \frac{1}{p}\ \mathrm{Var}(\cos(U_p(t))).$$

But $\mathrm{Var}(\cos(U)) = \mathbb{E}[\cos^2(U)] - (\mathbb{E}[\cos(U)])^2 = \mathbb{E}[\cos^2(U)]$, and Lemma 2.6 with $k = \ell = 1$ gives $\mathbb{E}[\cos^2(U)] = 1/2$. This yields the stated formula [3, 4]. □

We will frequently use the classical prime harmonic sum asymptotic (Mertens-type theorem),

$$\sum_{p \le y} \frac{1}{p} = \log\log y + B_1 + o(1) \qquad (y \to \infty), \tag{9}$$

for an absolute constant B_1; see [1, 2, 11]. In particular,

$$\sigma_y^2 = \frac{1}{2}\log\log y + O(1). \tag{10}$$

2.8. **The remainder $R_y(t)$ is uniformly controlled.** The higher-k modes are square-summable in p and therefore remain bounded (in L^2, hence in probability) as $y \to \infty$ [3, 4]. This is why S_y drives the leading Gaussian fluctuations.

Lemma 2.8 (Deterministic uniform bound for the remainder)**.** *For every $t \in \mathbb{R}$ and every realization of (θ_p),*

$$|R_y(t)| \;\le\; \sum_{p \le y}\sum_{k \ge 2} \frac{1}{k\,p^{k/2}} \;\le\; \sum_{p \le y}\sum_{k \ge 2} \frac{1}{p^{k/2}} \;=\; \sum_{p \le y} \frac{p^{-1}}{1 - p^{-1/2}}.$$

In particular, the series $\sum_p \frac{p^{-1}}{1-p^{-1/2}}$ converges, so $\sup_{y \ge 2}\sup_{t \in \mathbb{R}} |R_y(t)| < \infty$ [1, 2].

Proof. Use $|\cos(\cdot)| \le 1$ and drop the factor $1/k \le 1$:

$$|R_y(t)| \le \sum_{p \le y}\sum_{k \ge 2} \frac{1}{k\,p^{k/2}} \le \sum_{p \le y}\sum_{k \ge 2} \frac{1}{p^{k/2}} = \sum_{p \le y} \frac{p^{-1}}{1 - p^{-1/2}}.$$

Since $\frac{p^{-1}}{1-p^{-1/2}} \asymp p^{-1}$ for large p, convergence follows from $\sum_p p^{-1-\varepsilon}$ style comparison after expanding $1/(1 - p^{-1/2}) = 1 + O(p^{-1/2})$ [1, 2]; concretely,

$$\frac{p^{-1}}{1 - p^{-1/2}} = p^{-1} + O(p^{-3/2}),$$

and $\sum_p p^{-3/2} < \infty$ [1]. □

For the probabilistic theory later, an L^2 estimate is cleaner than a deterministic one [3, 4, 7].

Proposition 2.9 (L^2-boundedness of the remainder)**.** *For every $t \in \mathbb{R}$,*

$$\mathbb{E}[R_y(t)] = 0, \qquad \mathrm{Var}(R_y(t)) \le \frac{1}{2}\sum_{p\le y}\sum_{k\ge 2}\frac{1}{k^2\,p^k} \le C < \infty,$$

where C is an absolute constant independent of y and t. In particular, $R_y(t) = O_{L^2}(1)$ uniformly in y [3, 4, 7]*.*

Proof. Again write $U_p(t) = \theta_p - t\log p$; these are i.i.d. uniform. By Lemma 2.6, $\mathbb{E}[\cos(kU_p(t))] = 0$ for each $k \ge 2$, so $\mathbb{E}[R_y(t)] = 0$.

Decompose $R_y(t) = \sum_{p\le y} R_p(t)$ with

$$R_p(t) := \sum_{k\ge 2}\frac{\cos(kU_p(t))}{k\,p^{k/2}}.$$

The $R_p(t)$ are independent across primes. Hence

$$\mathrm{Var}(R_y(t)) = \sum_{p\le y}\mathrm{Var}(R_p(t)) = \sum_{p\le y}\mathbb{E}[R_p(t)^2],$$

since $\mathbb{E}[R_p(t)] = 0$. Expand the square:

$$\mathbb{E}[R_p(t)^2] = \mathbb{E}\left[\sum_{k\ge 2}\sum_{\ell\ge 2}\frac{\cos(kU_p(t))\cos(\ell U_p(t))}{k\ell\,p^{(k+\ell)/2}}\right].$$

Interchange sum and expectation (absolute convergence is easy because coefficients are summable), then apply Lemma 2.6: only $k = \ell$ contributes, and $\mathbb{E}[\cos^2(kU)] = 1/2$. Thus

$$\mathbb{E}[R_p(t)^2] = \sum_{k\ge 2}\frac{1}{k^2\,p^k}\cdot\frac{1}{2}.$$

Summing over $p \le y$ gives the first inequality. The double series $\sum_p\sum_{k\ge 2} k^{-2}p^{-k}$ converges absolutely because the $k = 2$ term already gives $\sum_p p^{-2} < \infty$, and higher k only improve convergence [1, 2]. This yields a finite absolute constant C. □

Corollary 2.10 (Remainder is negligible after Gaussian normalization)**.** *Let $\sigma_y^2 = \frac{1}{2}\sum_{p\le y}\frac{1}{p}$ as in* (8)*. Then, uniformly in $t \in \mathbb{R}$,*

$$\frac{R_y(t)}{\sigma_y} \xrightarrow[y\to\infty]{L^2} 0, \qquad \text{hence also} \qquad \frac{R_y(t)}{\sigma_y} \xrightarrow[y\to\infty]{\mathbb{P}} 0.$$

Proof. By Proposition 2.9, $\sup_{y,t}\mathbb{E}\big[R_y(t)^2\big] \le C < \infty$. On the other hand, (9) implies $\sigma_y^2 \to \infty$ as $y \to \infty$ [1, 2]. Therefore

$$\mathbb{E}\left[\left(\frac{R_y(t)}{\sigma_y}\right)^2\right] = \frac{\mathbb{E}[R_y(t)^2]}{\sigma_y^2} \le \frac{C}{\sigma_y^2} \xrightarrow[y\to\infty]{} 0,$$

uniformly in t. The L^2 convergence implies convergence in probability [3, 4]. □

Corollary 2.11 (Main term controls the asymptotic law)**.** *For each fixed $t \in \mathbb{R}$,*

$$\frac{X_y(t)}{\sigma_y} - \frac{S_y(t)}{\sigma_y} \xrightarrow[y\to\infty]{\mathbb{P}} 0.$$

In particular, any subsequential weak limit of $S_y(t)/\sigma_y$ is also a subsequential weak limit of $X_y(t)/\sigma_y$, and conversely.

Quantity	Mean	Variance / size
$S_y(t) = \sum_{p\le y} \cos(\theta_p - t\log p)/\sqrt{p}$	0	$\mathrm{Var}(S_y(t)) = \frac{1}{2}\sum_{p\le y}\frac{1}{p} = \sigma_y^2$
$R_y(t) = \sum_{p\le y}\sum_{k\ge 2} \cos(k(\theta_p - t\log p))/(kp^{k/2})$	0	$\mathrm{Var}(R_y(t)) \le C$ (uniform)
$X_y(t) = S_y(t) + R_y(t)$	0	$\mathrm{Var}(X_y(t)) = \sigma_y^2 + O(1)$

TABLE 1. First two moments and the main normalization scale. The Gaussian normalization is driven by $\sigma_y^2 \sim \frac{1}{2}\log\log y$ [1, 2].

Proof. This is immediate from $X_y(t) = S_y(t) + R_y(t)$ and Corollary 2.10. □

2.9. **Vector-valued viewpoint: isotropic complex fluctuations.** The random walk picture can be made algebraic [3, 4]. Recall

$$V_y(t) = \sum_{p\le y} \frac{e^{i(\theta_p - t\log p)}}{\sqrt{p}}, \qquad S_y(t) = \mathrm{Re}\ V_y(t).$$

Write also $T_y(t) := \mathrm{Im}\ V_y(t) = \sum_{p\le y} \sin(\theta_p - t\log p)/\sqrt{p}$.

Proposition 2.12 (Isotropy at second order)**.** *For every $t \in \mathbb{R}$,*

$$\mathbb{E}[V_y(t)] = 0, \qquad \mathbb{E}\big[|V_y(t)|^2\big] = \sum_{p\le y}\frac{1}{p},$$

and

$$\mathrm{Var}(S_y(t)) = \mathrm{Var}(T_y(t)) = \frac{1}{2}\sum_{p\le y}\frac{1}{p}, \qquad (S_y(t), T_y(t)) = 0.$$

Proof. As before, $U_p(t) := \theta_p - t\log p$ are i.i.d. uniform on $[0, 2\pi)$, so $\mathbb{E}[e^{iU_p(t)}] = 0$ and $\mathbb{E}[\cos(U_p(t))] = \mathbb{E}[\sin(U_p(t))] = 0$. Independence across primes gives $\mathbb{E}[V_y(t)] = 0$.

For the second moment,

$$\mathbb{E}\big[|V_y(t)|^2\big] = \mathbb{E}\Big[\sum_{p\le y}\sum_{q\le y}\frac{e^{i(U_p(t) - U_q(t))}}{\sqrt{pq}}\Big].$$

When $p \ne q$, independence gives $\mathbb{E}[e^{iU_p(t)}]\mathbb{E}[e^{-iU_q(t)}] = 0$, so only diagonal terms remain:

$$\mathbb{E}\big[|V_y(t)|^2\big] = \sum_{p\le y}\frac{1}{p}\,\mathbb{E}[|e^{iU_p(t)}|^2] = \sum_{p\le y}\frac{1}{p}.$$

Finally,

$$\mathrm{Var}(S_y(t)) = \sum_{p\le y}\frac{1}{p}\,\mathrm{Var}(\cos(U_p(t))) = \sum_{p\le y}\frac{1}{p}\cdot\frac{1}{2},$$

and similarly for $T_y(t)$ because $\mathbb{E}[\sin^2(U)] = 1/2$. Moreover,

$$(S_y(t), T_y(t)) = \sum_{p\le y}\frac{1}{p}\,\mathbb{E}\big[\cos(U_p(t))\sin(U_p(t))\big]$$

by independence, and $\mathbb{E}[\cos(U)\sin(U)] = (2\pi)^{-1}\int_0^{2\pi}\frac{1}{2}\sin(2u)\,du = 0$. □

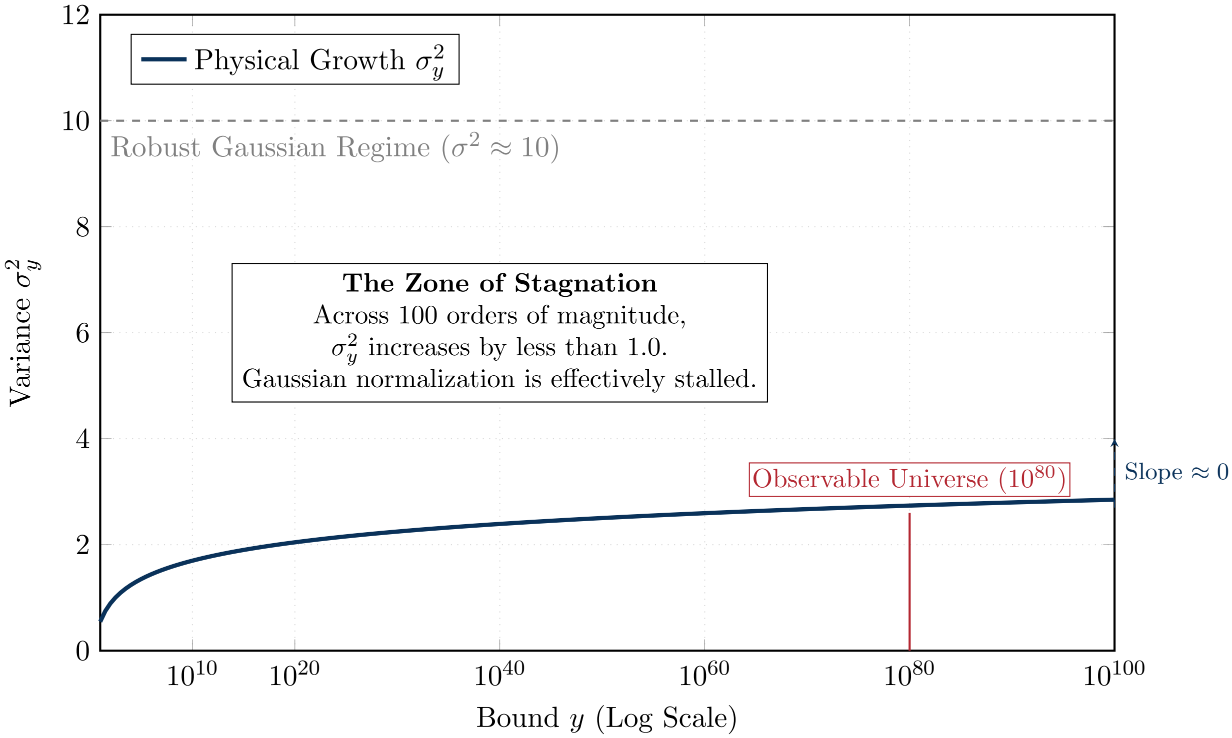

FIGURE 4. Schematic growth of σ_y^2. The function grows so slowly that even at the scale of the universe (red line), the variance is ≈ 2.6. To reach a variance of 10, y would need to be $\approx 10^{10^8}$, illustrating why the distribution $L(1,\chi)$ is rarely Gaussian in practice.

Remark 2.13 (Heuristic Gaussianity in two dimensions). Proposition 2.12 shows that $(S_y(t), T_y(t))$ is centered with covariance matrix $\sigma_y^2 I_2$. Since each prime contributes an independent bounded increment of size $p^{-1/2}$ and $\sum_{p\le y} p^{-1} \sim \log\log y$ [1, 2], a two-dimensional CLT suggests that

$$\frac{V_y(t)}{\sqrt{\sum_{p\le y} 1/p}} \approx \mathcal{N}_{\mathbb{C}}(0,1) \qquad \text{and} \qquad \frac{S_y(t)}{\sigma_y} \approx \mathcal{N}(0,1),$$

with quantitative error controlled by Lindeberg-type bounds (proved later) [3, 4, 7].

2.10. **A quantitative picture of the variance build-up (prime scale).** The normalization is slow: $\sigma_y^2 \sim \frac{1}{2}\log\log y$ [1, 2]. The following schematic highlights the double-logarithmic growth that makes these models subtle.

2.11. **What remains for the probabilistic core.** At this point we have reduced the leading behavior of $X_y(t)$ to the leading behavior of the main term $S_y(t)$:

$$\frac{X_y(t)}{\sigma_y} = \frac{S_y(t)}{\sigma_y} + o_{\mathbb{P}}(1).$$

The next sections will prove a central limit theorem (and then strengthen it quantitatively) for $S_y(t)/\sigma_y$. Conceptually, this is a CLT for a triangular array of independent, non-identically distributed bounded variables

$$\frac{\cos(\theta_p - t\log p)}{\sqrt{p}}, \qquad p \le y,$$

whose total variance is σ_y^2 [3, 4, 7].

Remark 2.14 (Preview of the next step). There are two standard routes.

First, one may verify Lindeberg's condition directly (it is easy here because each summand is bounded by $1/\sqrt{p}$), and then apply the Lindeberg–Feller theorem to conclude $S_y(t)/\sigma_y \Rightarrow \mathcal{N}(0,1)$ [3, 4, 7].

Second, one can compute the characteristic function by independence, expand $\log \mathbb{E}[\exp(i\lambda S_y/\sigma_y)]$ into a prime sum, and show the quadratic term dominates while higher cumulants are summable. This route is closer in spirit to Euler products and will also yield error terms [1, 2, 12]. We will take the second route, and keep the first as a quick cross-check.

3. CLT for the main term

3.1. **Variance scale.** Recall

$$S_y(t) = \sum_{p \le y} \frac{\cos(\theta_p - t\log p)}{\sqrt{p}}, \qquad U_p(t) := \theta_p - t\log p \pmod{2\pi},$$

so that the family $(U_p(t))_{p\le y}$ is i.i.d. uniform on $[0,2\pi)$ by Proposition 2.2. Define the variance parameter

$$\sigma_y^2 \;:=\; \operatorname{Var}\big(S_y(t)\big). \tag{11}$$

Proposition 3.1 (Explicit variance)**.** *For every $t \in \mathbb{R}$,*

$$\sigma_y^2 = \frac{1}{2}\sum_{p\le y}\frac{1}{p}. \tag{12}$$

In particular, σ_y^2 is increasing in y and $\sigma_y \to \infty$ as $y \to \infty$. Moreover, the prime harmonic sum satisfies

$$\sum_{p\le y}\frac{1}{p} = \log\log y + B_1 + o(1) \qquad (y\to\infty), \tag{13}$$

so $\sigma_y^2 = \frac{1}{2}\log\log y + O(1)$.

Proof. Write $S_y(t) = \sum_{p\le y} p^{-1/2}\cos(U_p(t))$. Independence across primes gives

$$\operatorname{Var}(S_y(t)) = \sum_{p\le y}\operatorname{Var}\left(\frac{\cos(U_p(t))}{\sqrt{p}}\right) = \sum_{p\le y}\frac{1}{p}\operatorname{Var}(\cos(U)),$$

where $U \sim \mathrm{Unif}[0,2\pi)$. Since $\mathbb{E}[\cos(U)] = 0$ and $\mathbb{E}[\cos^2(U)] = 1/2$ by Lemma 2.6, we have $\operatorname{Var}(\cos(U)) = 1/2$, yielding (12). The asymptotic (13) is a standard Mertens-type theorem for primes; see [1, 2, 11]. □

Remark 3.2 (What we actually need). For the central limit theorem, the only essential input is $\sigma_y^2 \to \infty$, i.e. $\sum_{p\le y} 1/p \to \infty$. The sharper $\log\log y$ growth is useful for interpretation, scaling comparisons, and numerics; see [1, 2].

3.2. **Lindeberg–Feller setup.** Define the prime-indexed triangular array

$$X_{p,y}(t) \;:=\; \frac{\cos(\theta_p - t\log p)}{\sqrt{p}}, \qquad p \le y,$$

so that

$$S_y(t) = \sum_{p\le y} X_{p,y}(t), \qquad \mathbb{E}[X_{p,y}(t)] = 0, \qquad \sum_{p\le y}\operatorname{Var}(X_{p,y}(t)) = \sigma_y^2.$$

Each $X_{p,y}(t)$ is bounded:

$$|X_{p,y}(t)| \le \frac{1}{\sqrt{p}} \le \frac{1}{\sqrt{2}}. \tag{14}$$

Lemma 3.3 (Uniform smallness of the largest summand after normalization)**.** *As $y \to \infty$,*

$$\max_{p\le y} \frac{|X_{p,y}(t)|}{\sigma_y} \;\longrightarrow\; 0,$$

uniformly in $t \in \mathbb{R}$.

Proof. By (14),

$$\max_{p\le y} \frac{|X_{p,y}(t)|}{\sigma_y} \le \frac{1}{\sigma_y\sqrt{2}}.$$

Since $\sigma_y \to \infty$ by Proposition 3.1, the right-hand side tends to 0. The bound does not depend on t. □

Lemma 3.4 (Lindeberg condition)**.** *For every $\varepsilon > 0$,*

$$\frac{1}{\sigma_y^2} \sum_{p\le y} \mathbb{E}\big[X_{p,y}(t)^2 \mathbf{1}\{|X_{p,y}(t)| > \varepsilon\sigma_y\}\big] \;\xrightarrow[y\to\infty]{}\; 0,$$

uniformly in $t \in \mathbb{R}$.

Proof. Fix $\varepsilon > 0$. By Lemma 3.3, for all sufficiently large y we have $|X_{p,y}(t)| \le \varepsilon\sigma_y$ for every prime $p \le y$ (uniformly in t). Hence the indicator $\mathbf{1}\{|X_{p,y}(t)| > \varepsilon\sigma_y\}$ is identically zero for all large y, and the whole sum vanishes. □

3.3. Central limit theorem.

Theorem 3.5 (CLT for the main term)**.** *Let $S_y(t)$ be as in* (5) *and let σ_y^2 be given by* (12)*. Then as $y \to \infty$,*

$$\frac{S_y(t)}{\sigma_y} \;\Rightarrow\; N(0,1),$$

uniformly in $t \in \mathbb{R}$ (in the sense that the limiting distribution does not depend on t).

Proof. We apply the Lindeberg–Feller central limit theorem to the independent triangular array $\{X_{p,y}(t)\}_{p\le y}$; see [3, 4, 7, 6]. The array is centered, the total variance is σ_y^2, and $\sigma_y^2 \to \infty$. Lemma 3.4 verifies Lindeberg's condition. Therefore

$$\frac{\sum_{p\le y} X_{p,y}(t)}{\sigma_y} \Rightarrow N(0,1),$$

which is exactly $S_y(t)/\sigma_y \Rightarrow N(0,1)$. □

Corollary 3.6 (CLT for $X_y(t)$)**.** *With $X_y(t) = S_y(t) + R_y(t)$ and $\sigma_y^2 = \frac{1}{2}\sum_{p\le y}\frac{1}{p}$, we have*

$$\frac{X_y(t)}{\sigma_y} \;\Rightarrow\; N(0,1), \qquad (y \to \infty).$$

Proof. By Corollary 2.10, $R_y(t)/\sigma_y \to 0$ in probability. By Theorem 3.5, $S_y(t)/\sigma_y \Rightarrow N(0,1)$. Slutsky's theorem yields $X_y(t)/\sigma_y \Rightarrow N(0,1)$; see [3, 4]. □

3.4. **Characteristic function expansion.** The Lindeberg proof is conceptually clean. For later quantitative refinements (Berry–Esseen type bounds), it is useful to record a parallel approach via characteristic functions, which makes the "Gaussian term" and "cumulant tail" explicit; see [5, 7, 6].

Lemma 3.7 (One-prime characteristic function expansion)**.** *Let $U \sim \mathrm{Unif}[0, 2\pi)$ and let $a \in \mathbb{R}$. Then*

$$\mathbb{E}\left[e^{ia\cos U}\right] = J_0(a),$$

where J_0 is the Bessel function of the first kind. Moreover, as $a \to 0$,

$$\log J_0(a) = -\frac{a^2}{4} + O(a^4).$$

Proof. The identity $\mathbb{E}[e^{ia\cos U}] = \frac{1}{2\pi}\int_0^{2\pi} e^{ia\cos u}\,du = J_0(a)$ is standard (Fourier–Bessel representation); see, for example, [5]. The Taylor expansion $J_0(a) = 1 - \frac{a^2}{4} + O(a^4)$ follows from expanding $e^{ia\cos u}$ and using $\mathbb{E}[\cos U] = 0$, $\mathbb{E}[\cos^2 U] = 1/2$, $\mathbb{E}[\cos^4 U] = 3/8$. Taking $\log(1-x) = -x + O(x^2)$ with $x = \frac{a^2}{4} + O(a^4)$ yields $\log J_0(a) = -\frac{a^2}{4} + O(a^4)$. □

Remark 3.8 (Why we mention J_0). Because our summands are $\cos(U_p(t))/\sqrt{p}$, the characteristic function of $S_y(t)$ factors into a product of $J_0(\lambda/(\sigma_y\sqrt{p}))$. The small-$a$ expansion above shows the Gaussian quadratic term dominates because $\sum_{p\le y} p^{-1}$ diverges while $\sum_{p\le y} p^{-2}$ converges; see [1, 2]. We will exploit this structure in the statistics/error-term section; see also the probabilistic-number-theory viewpoint in [12].

4. Remainder control and the CLT for X_y

4.1. **L^2-boundedness of the higher-order terms.** Recall the decomposition

$$X_y(t) = S_y(t) + R_y(t), \qquad S_y(t) = \sum_{p\le y} \frac{\cos(\theta_p - t\log p)}{\sqrt{p}}, \qquad R_y(t) = \sum_{p\le y}\sum_{k\ge 2} \frac{\cos(k(\theta_p - t\log p))}{k\,p^{k/2}}.$$

The key point is that the "$k \ge 2$" contribution is square-summable in p (already at $k = 2$), so it stays bounded while the variance of S_y grows like $\frac{1}{2}\log\log y$; see [1, 2] for the underlying prime-sum asymptotics and [3, 4] for the probabilistic scaling principles.

Lemma 4.1 (One-prime remainder is square-summable)**.** *Define, for each prime $p \le y$,*

$$R_p(t) := \sum_{k\ge 2} \frac{\cos(k(\theta_p - t\log p))}{k\,p^{k/2}}.$$

Then $\mathbb{E}[R_p(t)] = 0$ and

$$\mathrm{Var}(R_p(t)) = \mathbb{E}[R_p(t)^2] = \frac{1}{2}\sum_{k\ge 2}\frac{1}{k^2\,p^k} \le \frac{1}{2}\sum_{k\ge 2}\frac{1}{p^k} = \frac{1}{2}\cdot\frac{p^{-2}}{1-p^{-1}} \le \frac{1}{p^2},$$

uniformly in t.

Proof. Let $U_p(t) := \theta_p - t\log p \pmod{2\pi}$; as before $U_p(t)$ is uniform on $[0, 2\pi)$. Then $\mathbb{E}[\cos(kU_p(t))] = 0$ for $k \ge 1$ by Lemma 2.6, and hence $\mathbb{E}[R_p(t)] = 0$.

For the second-moment computation, expand and use Fourier orthogonality:

$$\mathbb{E}[R_p(t)^2] = \mathbb{E}\left[\sum_{k\ge 2}\sum_{\ell\ge 2}\frac{\cos(kU_p(t))\cos(\ell U_p(t))}{k\ell\,p^{(k+\ell)/2}}\right].$$

Interchange expectation and sums (absolute convergence is immediate since $\sum_{k\geq 2}(kp^{k/2})^{-1} < \infty$), and apply Lemma 2.6 to get

$$\mathbb{E}[\cos(kU)\cos(\ell U)] = \begin{cases} \frac{1}{2}, & k = \ell, \\ 0, & k \neq \ell. \end{cases}$$

Thus only diagonal terms remain:

$$\mathbb{E}[R_p(t)^2] = \frac{1}{2}\sum_{k\geq 2}\frac{1}{k^2\, p^k}.$$

Dropping $k^2 \geq 1$ yields

$$\mathbb{E}[R_p(t)^2] \leq \frac{1}{2}\sum_{k\geq 2}\frac{1}{p^k} = \frac{1}{2}\cdot\frac{p^{-2}}{1-p^{-1}} \leq \frac{1}{p^2},$$

since $1/(1-p^{-1}) \leq 2$ for all primes $p \geq 2$. □

Lemma 4.2 (The remainder is $O_{L^2}(1)$)**.** *Let $R_y(t)$ be as in* (6)*. Then* $\sup_{y\geq 2}\mathrm{Var}(R_y(t)) < \infty$*. In particular, $R_y(t) = O_{L^2}(1)$ uniformly in y and t, and hence $R_y(t) = o_{\mathbb{P}}(\sigma_y)$ as $y \to \infty$.*

Proof. Write $R_y(t) = \sum_{p\leq y} R_p(t)$ with $R_p(t)$ as in Lemma 4.1. Independence across primes (since the θ_p are independent) gives

$$\mathrm{Var}(R_y(t)) = \sum_{p\leq y}\mathrm{Var}(R_p(t)) \leq \sum_{p\leq y}\frac{1}{p^2} \leq \sum_{p}\frac{1}{p^2} < \infty,$$

which is uniform in y and t; see [1, 2] for standard comparisons with convergent prime sums.

Since $\sigma_y \to \infty$ by Proposition 3.1,

$$\mathbb{E}\left[\left(\frac{R_y(t)}{\sigma_y}\right)^2\right] = \frac{\mathrm{Var}(R_y(t))}{\sigma_y^2} \;\longrightarrow\; 0,$$

so $R_y(t)/\sigma_y \to 0$ in L^2 and therefore in probability; see [3, 4]. □

Remark 4.3 (Why this is the whole point of the model). The leading layer S_y carries variance $\sigma_y^2 = \frac{1}{2}\sum_{p\leq y} 1/p \sim \frac{1}{2}\log\log y$, whereas the $k \geq 2$ layers are square-summable in p and contribute only $O(1)$ fluctuations. This clean scale separation is one of the basic reasons random Euler products successfully predict Gaussian fluctuations at the $\sqrt{\log\log y}$ scale; see [1, 2, 12] for probabilistic-number-theory context.

4.2. **Main theorem.**

Theorem 4.4 (CLT for the random Euler-product log)**.** *Let $X_y(t) = \mathrm{Re}\log Z_y(t)$ be as in* (2)*, and let*

$$\sigma_y^2 = \frac{1}{2}\sum_{p\leq y}\frac{1}{p}.$$

Then as $y \to \infty$,

$$\frac{X_y(t)}{\sigma_y} \;\Rightarrow\; N(0,1).$$

Proof. Write $X_y(t) = S_y(t) + R_y(t)$. By Theorem 3.5, $S_y(t)/\sigma_y \Rightarrow N(0,1)$ (via Lindeberg–Feller; see [3, 4, 7]). By Lemma 4.2, $R_y(t)/\sigma_y \to 0$ in probability. Slutsky's theorem implies

$$\frac{X_y(t)}{\sigma_y} = \frac{S_y(t)}{\sigma_y} + \frac{R_y(t)}{\sigma_y} \;\Rightarrow\; N(0,1),$$

again with references for Slutsky in [3, 4]. □

5. Numerical experiments

We record one computational check (see Appendix C.1 for exact code): we simulate the normalized main term S_y/σ_y at a fixed cutoff y and compare its empirical distribution to $N(0,1)$ using a histogram with overlaid Gaussian density and a normal QQ-plot. This directly probes the conclusion of Theorem 3.5; see [3, 4] for standard diagnostics of Gaussian convergence.

Remark 5.1. By Proposition 2.2 we may fix $t = 0$ without loss, so we sample i.i.d. phases $\theta_p \sim \mathrm{Unif}[0, 2\pi)$ and compute $S_y = \sum_{p\le y} \cos(\theta_p)/\sqrt{p}$.

5.1. **Experimental setup.**

We generate primes up to y via a sieve, sample i.i.d. phases $\theta_p \sim \mathrm{Unif}[0, 2\pi)$, and compute

$$S_y = \sum_{p\le y} \frac{\cos\theta_p}{\sqrt{p}}, \qquad \sigma_y^2 = \mathrm{Var}(S_y) = \frac{1}{2}\sum_{p\le y}\frac{1}{p}, \qquad \frac{S_y}{\sigma_y}.$$

We repeat this for N_{trials} independent draws of the phase family $(\theta_p)_{p\le y}$ and plot the resulting samples of S_y/σ_y. The variance formula and the slow growth $\sigma_y^2 \sim \frac{1}{2}\log\log y$ follow from Mertens-type prime harmonic sum asymptotics; see [1, 2].

5.2. **Figures.**

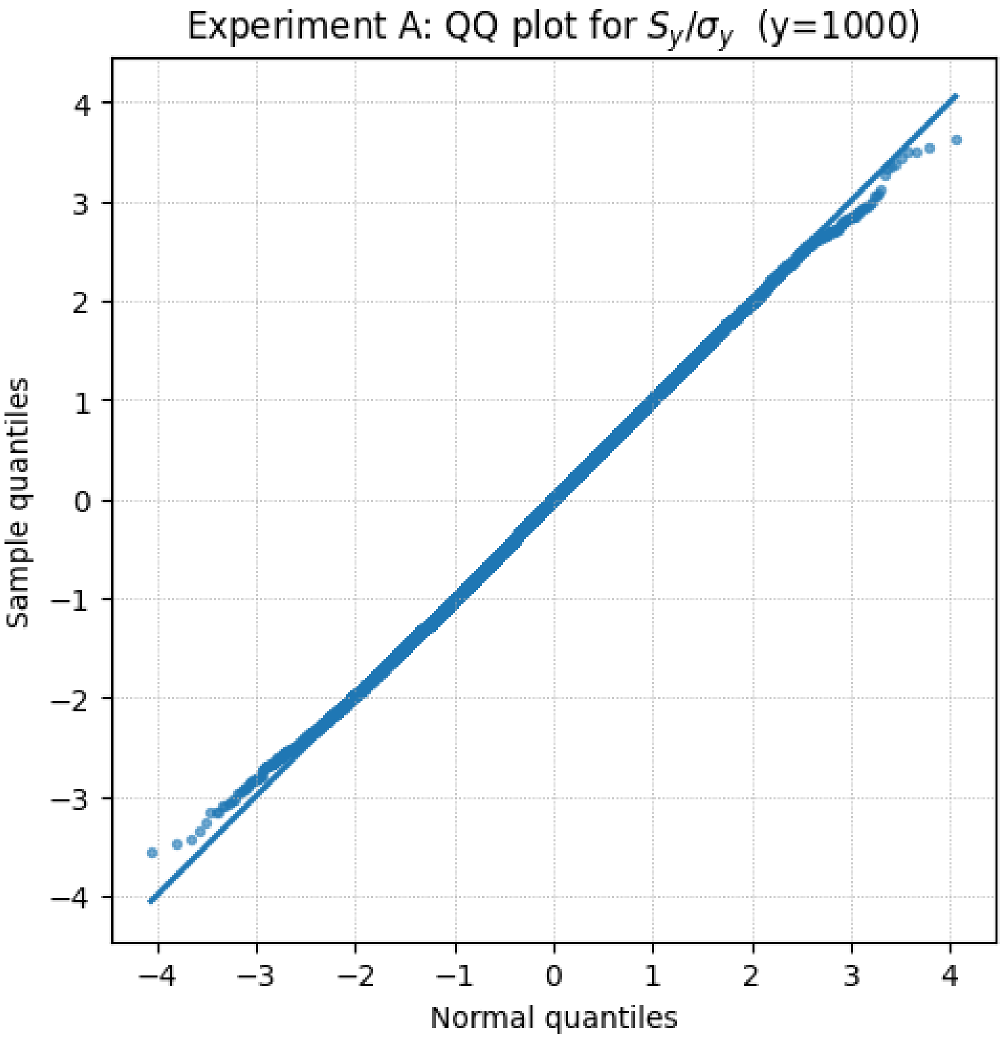

FIGURE 5. Histogram of S_y/σ_y for cutoff $y = 1000$ with the standard normal density overlaid. The agreement in the bulk is consistent with Theorem 3.5.

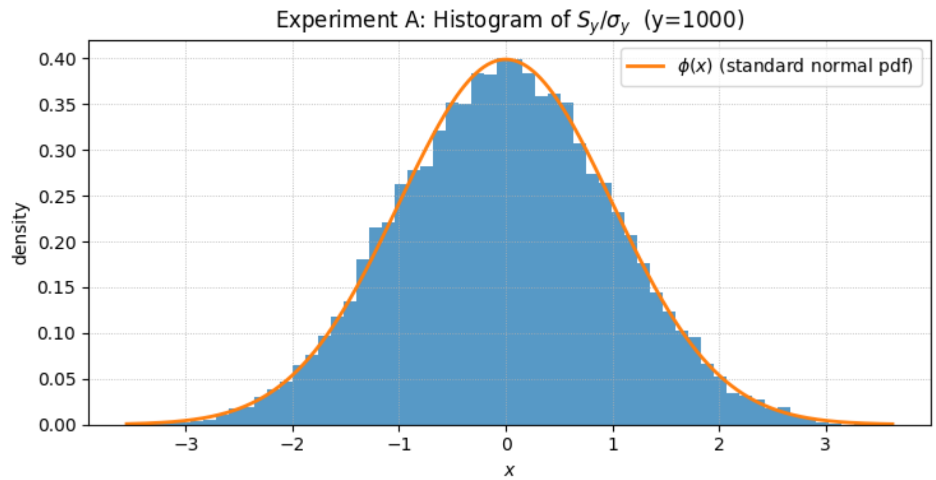

FIGURE 6. QQ plot of S_y/σ_y against a standard normal for cutoff $y = 1000$. Approximate linearity indicates Gaussian behavior; deviations in the extreme tails are expected at such modest cutoffs.

Reproducibility. The exact source code used to generate Figures 5–6 is recorded in Appendix C.1.

6. Discussion

6.1. **What the model captures (and what it omits).** The random Euler product isolates a single robust mechanism: writing $X_y(t) = \operatorname{Re}\log Z_y(t) = S_y(t) + R_y(t)$, the leading layer $S_y(t) = \sum_{p\le y}\cos(\theta_p - t\log p)/\sqrt{p}$ is a sum of independent, uniformly bounded increments with variance $\sigma_y^2 = \operatorname{Var}(S_y(t)) = \frac{1}{2}\sum_{p\le y} 1/p \sim \frac{1}{2}\log\log y$ by Mertens' theorem [1, 2]. Hence $S_y(t)/\sigma_y$ satisfies a Lindeberg–Feller CLT, while the higher layers $R_y(t) = \sum_{p\le y}\sum_{k\ge 2}\cos(k(\theta_p - t\log p))/(k\,p^{k/2})$ are uniformly bounded in L^2 and therefore $R_y(t) = o_{\mathbb{P}}(\sigma_y)$. This is exactly the input behind Theorem 4.4; see also [3, 4, 7] for the probabilistic limit-theorem framework.

What the model omits is arithmetic dependence. In the true Euler product for $\zeta(1/2+it)$ the phases p^{-it} are deterministic and coupled through a single parameter t, so one studies distribution in t rather than literal independence across primes; see [10] for background on $\zeta(s)$ and its classical analytic theory. The present model should be viewed as a mean-field surrogate that correctly reproduces the variance scale and typical Gaussian fluctuations in decorrelated regimes, but it is insensitive to structured resonance events, extreme tails, and global multi-t constraints; cf. [12] for a probabilistic-number-theory perspective.

Appendix A. Extra input

A.1. **Fourier facts on the circle.**

Lemma A.1 (Cosine orthogonality)**.** *Let $U \sim \operatorname{Unif}[0, 2\pi)$. For integers $k, \ell \ge 1$,*

$$\mathbb{E}[\cos(kU)] = 0, \qquad \mathbb{E}[\cos(kU)\cos(\ell U)] = \begin{cases} \frac{1}{2}, & k = \ell, \\ 0, & k \ne \ell. \end{cases}$$

Proof. $\mathbb{E}[\cos(kU)] = (2\pi)^{-1}\int_0^{2\pi}\cos(ku)\,du = 0$. Also $\cos(ku)\cos(\ell u) = \frac{1}{2}\cos((k-\ell)u) + \frac{1}{2}\cos((k+\ell)u)$, and integrating over $[0, 2\pi]$ kills every nonconstant cosine term. When $k = \ell$ the $(k-\ell)$ term is $\cos(0u) = 1$, contributing $1/2$. □

A.2. **A CLT tool (triangular arrays).**

Theorem A.2 (Lindeberg–Feller CLT, convenient form)**.** *Let $\{X_{n,k}\}$ be a triangular array of independent real random variables with $\mathbb{E}[X_{n,k}] = 0$ and $s_n^2 := \sum_k \operatorname{Var}(X_{n,k}) \to \infty$. Assume the Lindeberg condition holds: for every $\varepsilon > 0$,*

$$\frac{1}{s_n^2}\sum_k \mathbb{E}\big[X_{n,k}^2 \mathbf{1}\{|X_{n,k}| > \varepsilon s_n\}\big] \longrightarrow 0.$$

Then $\sum_k X_{n,k}/s_n \Rightarrow N(0,1)$.

Remark A.3. In our application $|X_{n,k}| \le a_{n,k}$ with $\max_k a_{n,k}/s_n \to 0$, which implies Lindeberg immediately because $|X_{n,k}| > \varepsilon s_n$ is eventually impossible for every fixed $\varepsilon > 0$.

A.3. **Prime harmonic sum (reference input).**

Theorem A.4 (Prime harmonic sum)**.** *As $y \to \infty$,*

$$\sum_{p\le y}\frac{1}{p} = \log\log y + B_1 + o(1),$$

for an absolute constant B_1.

Remark A.5. We only use divergence $\sum_{p\le y} 1/p \to \infty$ for the qualitative CLT, but the asymptotic $\sim \log\log y$ is useful for interpreting the normalization and numerics; see [1, 2].

Appendix B. Figures

B.1. The horizon of rationals: Ford circles. For a reduced rational $p/q \in \mathbb{Q}$ $(q \geq 1)$, the *Ford circle* $C_{p/q}$ is the circle in the upper half-plane tangent to $\mathbb{R}$ at $x = p/q$ with center and radius

$$\left(\frac{p}{q}, \frac{1}{2q^2}\right), \qquad \mathrm{rad}(C_{p/q}) = \frac{1}{2q^2}.$$

A fundamental property is the Farey tangency criterion:

$$C_{p/q} \text{ is tangent to } C_{r/s} \quad \Longleftrightarrow \quad |ps - rq| = 1,$$

so tangencies encode adjacency in the Farey graph and (equivalently) the combinatorics of continued fractions. The modular group $SL_2(\mathbb{Z})$ permutes the family $\{C_{p/q}\}$ via Möbius transformations.

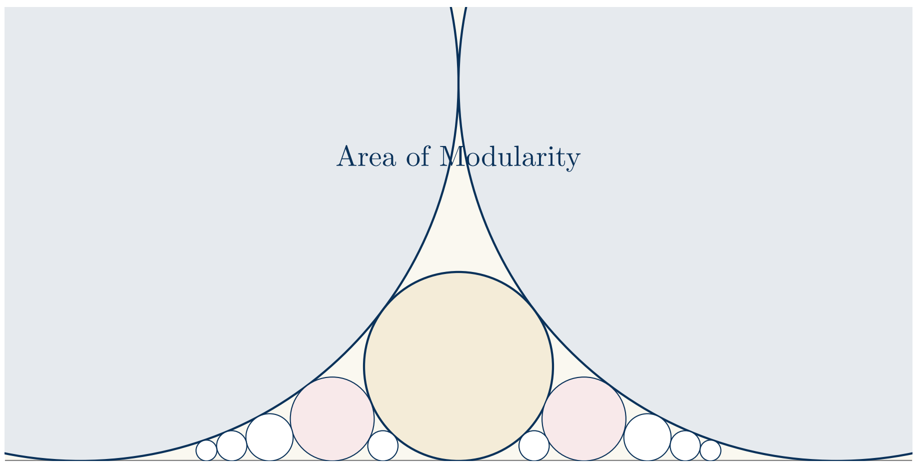

Figure 7. Ford circles in a bounded window. The circle $C_{p/q}$ has center $\left(\frac{p}{q}, \frac{1}{2q^2}\right)$ and radius $\frac{1}{2q^2}$. Two circles are tangent iff $|ps - rq| = 1$, i.e. p/q and r/s are Farey neighbors.

B.2. The dual skeleton: the Farey tessellation. The Farey tessellation is the ideal triangulation of the upper half-plane whose vertices are $\mathbb{Q} \cup \{\infty\}$ and whose edges are hyperbolic geodesics connecting Farey neighbors. In the upper half-plane model, these edges are semicircles orthogonal to $\mathbb{R}$ (together with vertical lines), and the action of $SL_2(\mathbb{Z})$ preserves the tessellation.

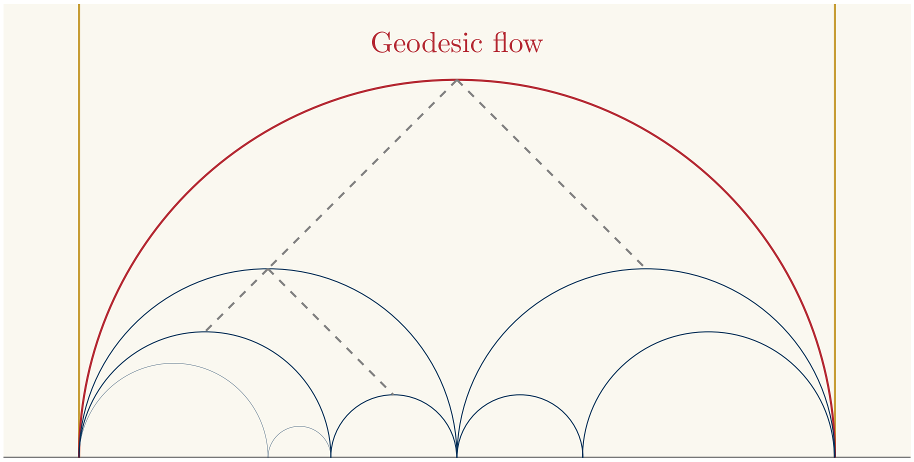

FIGURE 8. Farey tessellation in a bounded window. Geodesics (semicircles orthogonal to $\mathbb{R}$) connect Farey neighbors. The dashed gray lines indicate a schematic dual-tree structure.

APPENDIX C. SOURCE CODE

We record the exact Python code used to generate the figures in Section 5. Each script is self-contained and can be pasted into a local Python session or a notebook.

C.1. Experiment A: simulate S_y and check Gaussianity.

```
import numpy as np
import matplotlib.pyplot as plt

# ========================
# Experiment A:
# S_y(t) = sum_{p<=y} cos(theta_p - t log p)/sqrt(p)
# sigma_y^2 = (1/2) sum_{p<=y} 1/p
# Goal:
# Empirically check that S_y / sigma_y looks ~ N(0,1).
# Outputs:
# - Histogram + standard normal density overlay
# - QQ plot versus N(0,1)
# - A few summary diagnostics (mean, var, skew, kurtosis)
# ========================

# -------------------------
# 0) Prime utilities
# -------------------------
def primes_up_to(n: int) -> np.ndarray:
    """Return an array of all primes <= n (simple sieve)."""
    if n < 2:
        return np.array([], dtype=int)
    sieve = np.ones(n + 1, dtype=bool)
    sieve[:2] = False
    for p in range(2, int(n**0.5) + 1):
        if sieve[p]:
            sieve[p*p:n+1:p] = False
    return np.flatnonzero(sieve)

# -------------------------
# 1) Model: compute S_y samples
```

```
# ------------------------
def simulate_Sy_samples(y: int, nsamples: int, t: float = 0.0, seed: int = 0) -> tuple[np.
    ndarray, float]:
    """
    Simulate nsamples i.i.d. draws of S_y(t).
    Returns:
      Sy (nsamples,), sigma_y (float)
    """
    rng = np.random.default_rng(seed)
    ps = primes_up_to(y).astype(float)
    if ps.size == 0:
        raise ValueError("Need y >= 2 so there is at least one prime.")

    # Draw theta_p ~ Unif[0,2pi) independently for each sample and prime:
    # shape: (nsamples, nprimes)
    theta = rng.uniform(0.0, 2*np.pi, size=(nsamples, ps.size))

    # Deterministic phase shift -t log p (broadcast to samples)
    phase = theta - t * np.log(ps)[None, :]

    # Weighted cosine sum (broadcast 1/sqrt(p))
    weights = 1.0 / np.sqrt(ps)[None, :]
    Sy = np.sum(np.cos(phase) * weights, axis=1)

    # Theoretical sigma_y^2 = (1/2) sum_{p<=y} 1/p
    sigma2 = 0.5 * np.sum(1.0 / ps)
    sigma = float(np.sqrt(sigma2))
    return Sy, sigma

# ------------------------
# 2) Normal / QQ helpers (no SciPy)
# ------------------------
def normal_pdf(x: np.ndarray) -> np.ndarray:
    return (1.0 / np.sqrt(2*np.pi)) * np.exp(-0.5 * x**2)

def normal_ppf(p: np.ndarray) -> np.ndarray:
    """
    Approximate inverse CDF for N(0,1) using Peter J. Acklam's rational approximation.
    Accurate enough for QQ plots without SciPy.
    """
    # Coefficients (Acklam)
    a = np.array([
        -3.969683028665376e+01,
         2.209460984245205e+02,
        -2.759285104469687e+02,
         1.383577518672690e+02,
        -3.066479806614716e+01,
         2.506628277459239e+00
    ])
    b = np.array([
        -5.447609879822406e+01,
         1.615858368580409e+02,
        -1.556989798598866e+02,
         6.680131188771972e+01,
        -1.328068155288572e+01
    ])
    c = np.array([
```

```
        -7.784894002430293e-03,
        -3.223964580411365e-01,
        -2.400758277161838e+00,
        -2.549732539343734e+00,
         4.374664141464968e+00,
         2.938163982698783e+00
    ])
    d = np.array([
         7.784695709041462e-03,
         3.224671290700398e-01,
         2.445134137142996e+00,
         3.754408661907416e+00
    ])

    p = np.asarray(p)
    x = np.empty_like(p, dtype=float)

    plow = 0.02425
    phigh = 1.0 - plow

    # lower region
    mask = p < plow
    if np.any(mask):
        q = np.sqrt(-2.0 * np.log(p[mask]))
        x[mask] = (((((c[0]*q + c[1])*q + c[2])*q + c[3])*q + c[4])*q + c[5]) / \
                  ((((d[0]*q + d[1])*q + d[2])*q + d[3])*q + 1.0)

    # central region
    mask = (p >= plow) & (p <= phigh)
    if np.any(mask):
        q = p[mask] - 0.5
        r = q*q
        x[mask] = (((((a[0]*r + a[1])*r + a[2])*r + a[3])*r + a[4])*r + a[5]) * q / \
                  (((((b[0]*r + b[1])*r + b[2])*r + b[3])*r + b[4])*r + 1.0)

    # upper region
    mask = p > phigh
    if np.any(mask):
        q = np.sqrt(-2.0 * np.log(1.0 - p[mask]))
        x[mask] = -(((((c[0]*q + c[1])*q + c[2])*q + c[3])*q + c[4])*q + c[5]) / \
                   ((((d[0]*q + d[1])*q + d[2])*q + d[3])*q + 1.0)

    return x

def summarize_sample(x: np.ndarray) -> dict:
    x = np.asarray(x, dtype=float)
    mu = float(np.mean(x))
    var = float(np.var(x, ddof=0))
    sd = float(np.sqrt(var))
    # standardized moments
    z = (x - mu) / (sd + 1e-15)
    skew = float(np.mean(z**3))
    kurt = float(np.mean(z**4)) # (normal has 3)
    return {"mean": mu, "var": var, "sd": sd, "skew": skew, "kurtosis": kurt}

# ------------------------
# 3) Run + plots
```

```
# ------------------------
# Choose cutoffs (you can add more)
ys = [10**3, 10**4, 10**5] # increase if your runtime allows
nsamples = 20000 # increase for smoother QQ/hist
t = 0.0 # distribution does not depend on t
seed = 7

for y in ys:
    Sy, sigma = simulate_Sy_samples(y=y, nsamples=nsamples, t=t, seed=seed)
    Zy = Sy / sigma

    stats = summarize_sample(Zy)
    print(f"\n=== y={y} (#primes={len(primes_up_to(y))}) ===")
    print("sigma_y =", sigma)
    print("sample stats of S_y/sigma_y:",
          f"mean={stats['mean']:.4f}, var={stats['var']:.4f}, skew={stats['skew']:.4f}, kurt={
              stats['kurtosis']:.4f}")

    # ---- Histogram + N(0,1) overlay ----
    plt.figure(figsize=(7.5, 4))
    plt.title(rf"Experiment A: Histogram of $S_y/\sigma_y$ (y={y})")
    counts, bins, _ = plt.hist(Zy, bins=60, density=True, alpha=0.75, edgecolor="none")
    xs = np.linspace(bins[0], bins[-1], 400)
    plt.plot(xs, normal_pdf(xs), linewidth=2, label=r"$\phi(x)$ (standard normal pdf)")
    plt.xlabel(r"$x$")
    plt.ylabel("density")
    plt.legend()
    plt.grid(True, ls=":", lw=0.6)
    plt.tight_layout()
    plt.show()

    # ---- QQ plot vs N(0,1) ----
    plt.figure(figsize=(5.2, 5.2))
    plt.title(rf"Experiment A: QQ plot for $S_y/\sigma_y$ (y={y})")
    z_sorted = np.sort(Zy)
    # plotting positions
    p = (np.arange(1, nsamples + 1) - 0.5) / nsamples
    qn = normal_ppf(p)
    plt.scatter(qn, z_sorted, s=8, alpha=0.6)
    # 45-degree line
    lo = float(min(qn.min(), z_sorted.min()))
    hi = float(max(qn.max(), z_sorted.max()))
    plt.plot([lo, hi], [lo, hi], linewidth=2)
    plt.xlabel("Normal quantiles")
    plt.ylabel("Sample quantiles")
    plt.gca().set_aspect("equal", adjustable="box")
    plt.grid(True, ls=":", lw=0.6)
    plt.tight_layout()
    plt.show()
```

References

[1] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge Univ. Press, 2007.

[2] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, 3rd ed., Graduate Studies in Mathematics, American Mathematical Society, 2015.

[3] P. Billingsley, *Probability and Measure*, 3rd ed., Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, 1995.

[4] R. Durrett, *Probability: Theory and Examples*, 5th ed., Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge Univ. Press, 2019.

[5] W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. II*, 2nd ed., John Wiley & Sons, 1971.

[6] A. Gut, *Probability: A Graduate Course*, 2nd ed., Springer Texts in Statistics, Springer, 2013.

[7] V. V. Petrov, *Limit Theorems of Probability Theory: Sequences of Independent Random Variables*, Oxford Studies in Probability, Oxford Univ. Press, 1995.

[8] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, 3rd ed., Springer Texts in Statistics, Springer, 1997.

[9] G. Harman, *Prime-Detecting Sieves*, Princeton Univ. Press, 2007.

[10] A. Ivić, *The Riemann Zeta-Function: Theory and Applications*, Dover Publications, 2003.

[11] W. Narkiewicz, *The Development of Prime Number Theory: From Euclid to Hardy and Littlewood*, Springer Monographs in Mathematics, Springer, 2000.

[12] E. Kowalski, *An Introduction to Probabilistic Number Theory*, available as lecture notes / manuscript (various versions).