HOW THE CONFORMAL MODULUS OF AN ANNULUS QUANTITATIVELY CONTROLS HOLOMORPHIC MAPS

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ABSTRACT. Every doubly connected planar domain is conformally equivalent to a round annulus $A(r,R) = \{z \in \mathbb{C} : r < |z| < R\}$, and its conformal class is determined by the modulus

$$\operatorname{mod}(A) = \frac{1}{2\pi} \log\Big(\frac{R}{r}\Big).$$

We give a quantitative account of how $\operatorname{mod}(A)$ controls intrinsic geometry (extremal length, capacity, hyperbolic geometry) and the behavior of holomorphic maps (Fourier/Laurent damping on strip lifts, Schwarz–Pick distortion, proper mapping rigidity, and degeneration in the regimes $\operatorname{mod}(A) \to \infty$ and $\operatorname{mod}(A) \to 0$). On the round model, the hyperbolic density is explicit,

$$\rho_A(z) = \frac{\pi}{W} \cdot \frac{1}{|z|} \csc\Big(\frac{\pi}{W} \log\Big(\frac{|z|}{r}\Big)\Big), \qquad W = \log(R/r) = 2\pi \operatorname{mod}(A),$$

yielding derivative control for $f \in \operatorname{Hol}(A, \mathbb{D})$ via $\rho_{\mathbb{D}}(f(z))|f'(z)| \le \rho_A(z)$. We also show $\operatorname{Ext}_A(\Gamma_{\mathrm{rad}}) = \operatorname{mod}(A)$ and $⋒(A) = 1/\operatorname{mod}(A)$, and classify proper holomorphic maps between annuli: degree-n maps exist exactly when $\operatorname{mod}(A') = n \operatorname{mod}(A)$, in which case $f(z) = c\, z^{\pm n}$ in round coordinates.

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1. Introduction

Annuli are the simplest planar domains whose conformal geometry is not rigid. A round annulus is

$$A(r,R) = \{z \in \mathbb{C} : r < |z| < R\}, \qquad 0 < r < R < \infty,$$

and its conformal modulus is

$$\mathrm{mod}(A(r,R)) := \frac{1}{2\pi} \log\Big(\frac{R}{r}\Big).$$

For an arbitrary doubly connected domain $A \subset \mathbb{C}$ with two nondegenerate boundary components, $\mathrm{mod}(A)$ is defined by conformal invariance via a conformal equivalence $A \simeq A(r,R)$; existence and uniqueness of such a normal form are classical (see, e.g., [2, 3, 4, 47]). The point of this paper is that, once one works intrinsically (extremal length/capacity, hyperbolic geometry, strip lifts), many sharp analytic and geometric statements on annuli admit constants depending on the domain only through $\mathrm{mod}(A)$ (and, when topological winding is unavoidable, through the integer induced on $\pi_1(A) \cong \mathbb{Z}$). The guiding mechanism is the logarithmic covering:

$$z = e^w : \quad S_{r,R} := \{w \in \mathbb{C} : \log r < \Re w < \log R\} \longrightarrow A(r,R), \qquad w \sim w + 2\pi i,$$

so analysis on A becomes periodic analysis on a strip of width $\log(R/r) = 2\pi\,\mathrm{mod}(A)$; Fourier modes in $\Im w$ become Laurent modes in z, and long/short cylinder limits correspond to $\mathrm{mod}(A) \to \infty$ and $\mathrm{mod}(A) \to 0$ (cf. Sections 4, 8). This one-parameter geometry makes annuli a clean testbed for quantitative questions about boundary interaction, derivative distortion, rigidity of proper maps, and degeneration of families under modulus limits.

1.1. **Annuli as the first non-simply-connected testbed.**

By the Riemann mapping theorem, simply connected proper domains are conformally equivalent to $\mathbb{D}$, so (after passing to intrinsic metrics) the geometry has no continuous shape parameter. Doubly connected domains are the first setting where a continuous conformal invariant appears: every annulus-like domain is conformally equivalent to a round annulus, and the equivalence class is determined by $\mathrm{mod}(A) \in (0,\infty)$ [2, 47]. In logarithmic coordinates, the fundamental rectangle has height 2π and width $W = \log(R/r)$, and the Euclidean aspect ratio $W/(2\pi)$ is exactly $\mathrm{mod}(A)$. Informally, $\mathrm{mod}(A) \gg 1$ corresponds to a long cylinder (thin annulus), while $\mathrm{mod}(A) \ll 1$ corresponds to a short cylinder (thick annulus); the sections below turn this slogan into explicit formulas and sharp inequalities.

1.2. **Main results.**

We record representative quantitative statements proved later. In each case we include a brief proof idea and a pointer to the section where the complete argument appears.

Theorem 1.1 (Extremal length characterizations of modulus)**.** *Let A be an annulus. Let Γ_{rad} be the family of rectifiable curves in A connecting the two boundary components, and let Γ_{sep} be the family of rectifiable closed curves in A separating the boundary components. Then*

$$\mathrm{Ext}_A(\Gamma_{\mathrm{rad}}) = \mathrm{mod}(A), \qquad \mathrm{Ext}_A(\Gamma_{\mathrm{sep}}) = \frac{1}{\mathrm{mod}(A)}, \qquad \mathrm{Ext}_A(\Gamma_{\mathrm{rad}})\,\mathrm{Ext}_A(\Gamma_{\mathrm{sep}}) = 1.$$

Proof sketch. Reduce to the round model, pull back to the cylinder/strip, and compute the extremal metric explicitly for Γ_{rad}; duality then gives the reciprocal identity for Γ_{sep}. Full details appear in Section 3; see also the standard extremal-length treatments in [2, 47].

Theorem 1.2 (Capacity formula and extremizer)**.** *For the round annulus $A(r,R)$, the condenser capacity between boundary components is*

$$\Cap\big(A(r,R)\big) = \frac{2\pi}{\log(R/r)} = \frac{1}{\mathrm{mod}(A(r,R))}.$$

The energy-minimizing harmonic function with boundary values 0 *on* $|z| = r$ *and* 1 *on* $|z| = R$ *is*

$$u(z) = \frac{\log|z| - \log r}{\log R - \log r}.$$

Proof sketch. Solve the two-boundary Dirichlet problem in logarithmic radius, compute the Dirichlet energy, and identify the minimizer by convexity/uniqueness of harmonic solutions. See Section 3 (and compare [2, 47]).

Theorem 1.3 (Proper holomorphic maps and modulus scaling). *Let* $A = A(r, R)$ *and* $A' = A(r', R')$ *be round annuli. If* $f : A \to A'$ *is a proper holomorphic map of (topological) degree* $n \in$, *then after composing with rotations (and possibly inversion* $z \mapsto 1/z$*),*

$$f(z) = c\, z^n \qquad \text{for some } c \in \mathbb{C} \setminus \{0\},$$

and necessarily

$$\operatorname{mod}(A') = n \operatorname{mod}(A).$$

Proof sketch. Lift f to a holomorphic map between logarithmic strips; the deck equivariance encodes $\deg(f)$ as a vertical shift. Subtracting the linear part yields a periodic holomorphic function with bounded real part, hence constant, forcing an affine lift and therefore a monomial downstairs. The modulus scaling is then immediate from boundary radii. See Section 7; background on proper maps and coverings can be found in [4, 3, 2].

Theorem 1.4 (Schwarz–Pick control with explicit modulus dependence). *Let* A *be an annulus equipped with its hyperbolic density* ρ_A*, and let* $f \in \operatorname{Hol}(A, \mathbb{D})$*. Then for all* $z \in A$*,*

$$\rho_{\mathbb{D}}\big(f(z)\big)\, |f'(z)| \le \rho_A(z).$$

Moreover, in the round model $A(r, R)$ *the density* ρ_A *has an explicit formula and sharp comparison bounds in terms of* $W = \log(R/r) = 2\pi \operatorname{mod}(A)$ *and a logarithmic boundary-distance scale, yielding explicit Euclidean derivative bounds with constants depending only on* $\operatorname{mod}(A)$.

Proof sketch. Schwarz–Pick is proved by lifting to universal covers and applying the disk Schwarz lemma; the modulus enters through an explicit computation of ρ_A via the strip-to-half-plane map and through comparison with a cylindrical quasihyperbolic density. See Sections 5 and 6; standard references for Schwarz–Pick and hyperbolic metrics include [3, 12].

Theorem 1.5 (Fourier/Laurent mode decay controlled by strip width). *Let* $A = A(r, R)$ *and set* $W = \log(R/r) = 2\pi \operatorname{mod}(A)$*. If* f *is holomorphic on* A *and bounded by* M*, and*

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

is its Laurent expansion, then the coefficients satisfy an explicit exponential suppression in $|n|W$ *(equivalently, in* $|n| \operatorname{mod}(A)$*). In the normalized annulus* $A(q, 1)$*, one obtains a clean bound of the form*

$$|a_n| \le M\, q^{|n|/2} = \exp(-\pi |n| \operatorname{mod}(A)).$$

Proof sketch. Lift to the strip and extract Fourier coefficients by contour shifting across the strip width; holomorphy on a wider strip forces exponential decay of Fourier modes, which become Laurent modes under $z = e^w$. See Section 4; compare classical Laurent theory in [3].

Theorem 1.6 (Degeneration and normal families in modulus regimes). *Let* $\{A_k\}$ *be annuli with* $\operatorname{mod}(A_k) \to \infty$*, and let* $f_k \in \operatorname{Hol}(A_k, \mathbb{D})$*. After conformal identification of* A_k *with normalized cylinders/annuli, one obtains subsequential compactness on fixed subannuli, and on the mid-cylinder the functions become increasingly angularly rigid, with quantitative rates dictated by* $\operatorname{mod}(A_k)$*. A complementary blow-up of intrinsic scales occurs as* $\operatorname{mod}(A_k) \to 0$.

Proof sketch. Combine Montel normality on fixed subannuli with the modulus-dependent Fourier/Laurent suppression on long cylinders; the thick regime follows from the explicit ρ_A prefactor $\pi/W \sim 1/\mathrm{mod}(A)$. See Section 8.

2. Model annuli, normalization, and modulus

2.1. Round annuli and conformal symmetries.

Definition 2.1 (Round annulus)**.** For numbers $0 < r < R < \infty$, the *round annulus* with radii (r, R) is

$$A(r, R) := \{z \in \mathbb{C} : r < |z| < R\}.$$

Its boundary components are the circles $\{|z| = r\}$ and $\{|z| = R\}$.

Remark 2.2 (Two basic operations). Two elementary conformal operations preserve the annulus shape in the sense of biholomorphic equivalence. Scaling by a nonzero complex number sends $A(r, R)$ to another round annulus: if $\lambda \in \mathbb{C} \setminus \{0\}$ then $z \mapsto \lambda z$ maps $A(r, R)$ biholomorphically onto $A(|\lambda|r, |\lambda|R)$. Inversion swaps the boundary circles: $z \mapsto 1/z$ is biholomorphic $A(r, R) \to A(1/R, 1/r)$. These two symmetries already indicate that any conformal invariant of $A(r, R)$ should depend only on the ratio R/r. For standard background on such invariance principles in planar conformal geometry, see [1, 10, 3].

Lemma 2.3 (Connectivity and fundamental group)**.** *The round annulus $A(r, R)$ is doubly connected and $\pi_1(A(r, R)) \cong \mathbb{Z}$.*

Proof. In the Riemann sphere $\widehat{\mathbb{C}}$ the complement of $A(r, R)$ is

$$\widehat{\mathbb{C}} \setminus A(r, R) = \{|z| \le r\} \;\cup\; \big(\{|z| \ge R\} \cup \{\infty\}\big),$$

and these two sets are the two connected components of the complement. This is exactly the definition of a doubly connected domain.

To compute the fundamental group, we exhibit an explicit deformation retraction onto a circle. Let $\rho_0 := \sqrt{rR}$ and define

$$H : [0, 1] \times A(r, R) \to A(r, R), \qquad H(t, z) = \Big(\frac{\rho_0}{|z|}\Big)^t z.$$

When $t = 0$ this is the identity map. When $t = 1$ we have

$$|H(1, z)| = \Big(\frac{\rho_0}{|z|}\Big)|z| = \rho_0,$$

so $H(1, z)$ lies on the circle $\{|z| = \rho_0\}$. If $|z| = \rho_0$, then $(\rho_0/|z|)^t = 1$ for all t and therefore $H(t, z) = z$ for the entire homotopy.

It remains to check that $H(t, z)$ stays inside the annulus whenever $r < |z| < R$ and $t \in [0, 1]$. For such z we compute

$$|H(t, z)| = (\rho_0)^t |z|^{1-t}.$$

The quantity $(\rho_0)^t|z|^{1-t}$ is the geometric interpolation between $|z|$ and ρ_0. Since both $|z|$ and ρ_0 lie strictly between r and R, it follows that $|H(t, z)|$ also lies strictly between r and R. Hence H is a deformation retraction of $A(r, R)$ onto the circle $\{|z| = \rho_0\} \cong S^1$. Consequently,

$$\pi_1(A(r, R)) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

This computation is standard in discussions of annuli as doubly connected domains; see [4, 3]. ☐

2.2. **The modulus of a round annulus and its invariances.**

Definition 2.4 (Modulus of a round annulus)**.** For $A(r, R)$ we define its conformal modulus by

$$\operatorname{mod}\bigl(A(r, R)\bigr) := \frac{1}{2\pi} \log\Bigl(\frac{R}{r}\Bigr).$$

Remark 2.5 (Choice of normalization). The quantity $\log(R/r)$ is forced by scaling and inversion, while the factor $1/(2\pi)$ is chosen so that the modulus becomes the Euclidean aspect ratio in logarithmic coordinates and so that it matches the extremal length and capacity normalizations used later. These conventions go back to classical sources and are standard in modern treatments; see [1, 18, 21].

Proposition 2.6 (Elementary invariances)**.** *For any* $0 < r < R < \infty$ *and any* $\lambda \in \mathbb{C} \setminus \{0\}$*,*

$$\operatorname{mod}(A(|\lambda|r, |\lambda|R)) = \operatorname{mod}(A(r, R)), \qquad \operatorname{mod}(A(1/R, 1/r)) = \operatorname{mod}(A(r, R)).$$

Equivalently, $\operatorname{mod}(A(r, R))$ *depends only on the ratio* R/r*.*

Proof. Using the definition,

$$\operatorname{mod}(A(|\lambda|r, |\lambda|R)) = \frac{1}{2\pi} \log\Bigl(\frac{|\lambda|R}{|\lambda|r}\Bigr) = \frac{1}{2\pi} \log\Bigl(\frac{R}{r}\Bigr) = \operatorname{mod}(A(r, R)),$$

where the cancellation of $|\lambda|$ occurs inside the logarithm. For inversion,

$$\operatorname{mod}(A(1/R, 1/r)) = \frac{1}{2\pi} \log\Bigl(\frac{1/r}{1/R}\Bigr) = \frac{1}{2\pi} \log\Bigl(\frac{R}{r}\Bigr) = \operatorname{mod}(A(r, R)).$$

□

Remark 2.7 (A particularly convenient normalization). Because scaling does not change modulus, it is often convenient to normalize the outer radius to 1. Writing $q := r/R \in (0, 1)$, scaling by $1/R$ gives a biholomorphism $A(r, R) \cong A(q, 1)$. In these normalized coordinates,

$$\operatorname{mod}(A(q, 1)) = \frac{1}{2\pi} \log\Bigl(\frac{1}{q}\Bigr).$$

Equivalently, if $m = \operatorname{mod}(A)$ then $q = e^{-2\pi m}$. We will use the geometric parameters (r, R), the ratio q, and the modulus m interchangeably, depending on which makes a later estimate most transparent; compare [1, 10].

2.3. **Logarithmic coordinates and the strip/cylinder model.** This subsection makes the modulus appear as a literal Euclidean width via the coordinate $w = \log z$.

Definition 2.8 (The logarithmic strip associated to $A(r, R)$)**.** For $0 < r < R$, define the open vertical strip

$$S_{r,R} := \{w \in \mathbb{C} : \log r < \Re w < \log R\}.$$

Proposition 2.9 (Exponential covering of the annulus)**.** *Define* $\exp : \mathbb{C} \to \mathbb{C}\setminus\{0\}$ *by* $\exp(w) = e^w$*. Then the restriction*

$$\exp : S_{r,R} \longrightarrow A(r, R)$$

is a holomorphic covering map. The deck group is generated by the translation $T(w) = w + 2\pi i$*. Moreover the strip width equals*

$$\log R - \log r = \log(R/r) = 2\pi \operatorname{mod}(A(r, R)).$$

Proof. Let $w \in S_{r,R}$ and write $w = u + iv$ with $u = \Re w$. Then

$$|\exp(w)| = |e^{u+iv}| = |e^u e^{iv}| = e^u.$$

Since $\log r < u < \log R$, exponentiating gives $r < e^u < R$, hence $\exp(w) \in A(r,R)$.

Conversely, take $z \in A(r,R)$ and write $z = \rho e^{i\theta}$ with $\rho \in (r,R)$. Choose any real θ satisfying $e^{i\theta} = z/|z|$ and set $w := \log\rho + i\theta$. Then $\Re w = \log\rho$ lies strictly between $\log r$ and $\log R$, so $w \in S_{r,R}$ and $e^w = z$. Thus $\exp(S_{r,R}) = A(r,R)$.

The map exp is entire and satisfies $\exp'(w) = e^w \neq 0$ for all w, so it is locally biholomorphic on $S_{r,R}$. To identify the covering fibers, assume $e^{w_1} = e^{w_2}$. Then $e^{w_1 - w_2} = 1$. Writing $w_1 - w_2 = a + ib$, we obtain $1 = e^{a+ib} = e^a(\cos b + i\sin b)$. Taking absolute values gives $e^a = 1$ so $a = 0$, and then $e^{ib} = 1$ forces $b \in 2\pi\mathbb{Z}$. Therefore $w_1 - w_2 = 2\pi i k$ for some $k \in \mathbb{Z}$, which shows the deck group is generated by $T(w) = w + 2\pi i$.

Finally, the Euclidean width of the strip is $(\log R) - (\log r) = \log(R/r)$, and by Definition 2.4 this equals $2\pi \operatorname{mod}(A(r,R))$. The covering-space viewpoint for exp and its relation to annuli is standard; see [4, 2, 3]. □

Remark 2.10 (Cylinder/rectangle picture). A fundamental domain for the deck action is the rectangle

$$\mathcal{R}_{r,R} := \{w \in \mathbb{C} : \log r < \Re w < \log R,\ 0 < \Im w < 2\pi\}.$$

The identification is by the vertical translation $w \sim w + 2\pi i$, so it is the horizontal edges $\Im w = 0$ and $\Im w = 2\pi$ that are glued. After this identification, $\mathcal{R}_{r,R}$ becomes a Euclidean cylinder of circumference 2π and length $\log(R/r)$, and its aspect ratio is

$$\frac{\log(R/r)}{2\pi} = \operatorname{mod}(A(r,R)).$$

This cylinder model is a convenient geometric mnemonic and is used frequently in the literature; see [1, 4].

2.4. **Modulus for general annuli and well-definedness.** We now extend the definition beyond round annuli. The relevant class is that of nondegenerate doubly connected domains.

Definition 2.11 (Annulus, nondegenerate)**.** A domain $A \subset \mathbb{C}$ is called an *annulus* (or *doubly connected domain*) if $\widehat{\mathbb{C}} \setminus A$ has exactly two connected components. It is *nondegenerate* if both complementary components contain more than one point (equivalently, neither boundary component collapses to a point).

Theorem 2.12 (Conformal classification of nondegenerate annuli)**.** *Let $A \subset \mathbb{C}$ be a nondegenerate annulus. Then there exist numbers $0 < r < R < \infty$ and a biholomorphism*

$$\phi : A \longrightarrow A(r,R).$$

Moreover, the number $\operatorname{mod}(A(r,R))$ *is uniquely determined by A: if also $A \cong A(r',R')$ biholomorphically, then*

$$\frac{1}{2\pi}\log\Big(\frac{R}{r}\Big) = \frac{1}{2\pi}\log\Big(\frac{R'}{r'}\Big).$$

Proof. A classical proof constructs a canonical harmonic function separating the boundary components and then builds a single-valued holomorphic coordinate by exponentiating a normalized harmonic conjugate on the universal cover. This approach appears in standard references; see [2, 4, 18, 27].

Let K_0 and K_1 be the two connected components of $\widehat{\mathbb{C}} \setminus A$. Nondegeneracy ensures each K_j contains at least two points. Solve the Dirichlet problem on A with boundary data 0 on the boundary component adjacent to K_0 and 1 on the boundary component adjacent to K_1, and let

$u : A \to \mathbb{R}$ denote the resulting harmonic function. The maximum principle implies $0 < u < 1$ throughout A, so u is nonconstant and its level sets separate the boundary components.

Write $z = x + iy$ and express $du = u_x\,dx + u_y\,dy$. Define the conjugate 1-form

$$\omega := -u_y\,dx + u_x\,dy.$$

If v is any function with $dv = \omega$, then the identities $v_x = -u_y$ and $v_y = u_x$ hold, which are precisely the Cauchy–Riemann equations for $F = u + iv$, so F is holomorphic wherever such v exists. A direct computation using harmonicity shows ω is closed: starting from $\omega = -u_y\,dx + u_x\,dy$,

$$d\omega = -(u_{yx}\,dx+u_{yy}\,dy)\wedge dx+(u_{xx}\,dx+u_{xy}\,dy)\wedge dy = -(u_{yy}\,dy\wedge dx)+(u_{xx}\,dx\wedge dy) = (u_{xx}+u_{yy})\,dx\wedge dy = 0.$$

Because A is not simply connected, ω need not be exact on A. Fix a positively oriented loop γ generating $\pi_1(A) \cong \mathbb{Z}$ and define its period

$$P := \int_\gamma \omega.$$

If γ_1 and γ_2 are two positively oriented generator loops, they are homotopic in A. For a homotopy $H : [0,1] \times [0,1] \to A$ with $H(0,\cdot) = \gamma_1$, $H(1,\cdot) = \gamma_2$, and fixed endpoints, Stokes' theorem gives

$$\int_{\partial([0,1]\times[0,1])} H^*\omega = \int_{[0,1]\times[0,1]} H^*(d\omega) = 0,$$

and the contributions from the constant vertical sides vanish, leaving $\int_{\gamma_1}\omega = \int_{\gamma_2}\omega$. Thus P depends only on the positive generator class. The function u is not constant, and one shows $P \neq 0$; after choosing the positive orientation, we take $P > 0$. For the round annulus one can compute explicitly that $P = 2\pi/\log(R/r)$, and the general case follows from the same flux interpretation together with standard boundary-point lemmas in potential theory; see [27, 25, 18].

Pass to the universal cover $\pi : \widetilde{A} \to A$. Since $\widetilde{A}$ is simply connected and $d(\pi^*\omega) = \pi^*(d\omega) = 0$, the pullback $\widetilde{\omega} := \pi^*\omega$ is exact, so there exists a function $\widetilde{v}$ with $d\widetilde{v} = \widetilde{\omega}$. Let $\widetilde{u} := u \circ \pi$ and set $\widetilde{F} := \widetilde{u} + i\widetilde{v}$, which is holomorphic on $\widetilde{A}$. Under a deck transformation τ corresponding to one positive loop around the hole, $\widetilde{u}$ is unchanged while $\widetilde{v}$ increases by P, so $\widetilde{F}(\tau\tilde{z}) = \widetilde{F}(\tilde{z}) + iP$.

Define

$$\widetilde{\Phi}(\tilde{z}) := \exp\Big(\frac{2\pi}{P}\,\widetilde{F}(\tilde{z})\Big).$$

Then

$$\widetilde{\Phi}(\tau\tilde{z}) = \exp\Big(\frac{2\pi}{P}\big(\widetilde{F}(\tilde{z}) + iP\big)\Big) = \widetilde{\Phi}(\tilde{z})\exp(2\pi i) = \widetilde{\Phi}(\tilde{z}),$$

so $\widetilde{\Phi}$ is deck-invariant and descends to a holomorphic map $\Phi : A \to \mathbb{C} \setminus \{0\}$ with $\Phi \circ \pi = \widetilde{\Phi}$. Its modulus is controlled by u:

$$|\Phi(z)| = \exp\Big(\frac{2\pi}{P}u(z)\Big),$$

so the boundary component where $u = 0$ maps to $|w| = 1$, and the boundary component where $u = 1$ maps to $|w| = e^{2\pi/P}$. Since $0 < u < 1$ in A, we obtain $1 < |\Phi(z)| < e^{2\pi/P}$, hence $\Phi(A) \subset A(1, e^{2\pi/P})$. Because Φ is nonconstant holomorphic, it is open, and because its boundary behavior approaches both boundary circles, one concludes $\Phi(A) = A(1, e^{2\pi/P})$.

To see that Φ is a biholomorphism, it suffices to observe that a generator loop γ winds once around the hole, and along such a loop the change in the argument of Φ is 2π: the change in $\widetilde{v}$ along a lift is P, and the prefactor $2\pi/P$ turns this into an argument change of 2π. Thus $\Phi_* : \pi_1(A) \to \pi_1(A(1, e^{2\pi/P})) \cong \mathbb{Z}$ is multiplication by 1. In the theory of proper holomorphic maps between annuli, the induced map on π_1 equals the covering degree, so degree one forces Φ to be a biholomorphism; we will justify this systematically in Section 7, and standard references discuss this covering viewpoint as well [4, 1].

Uniqueness of the modulus follows immediately from the explicit target annulus:

$$\operatorname{mod}\bigl(A(1, e^{2\pi/P})\bigr) = \frac{1}{2\pi}\log(e^{2\pi/P}) = \frac{1}{P}.$$

Since P was determined intrinsically from A, the modulus is uniquely determined by A. □

Definition 2.13 (Modulus of a general annulus)**.** Let $A \subset \mathbb{C}$ be a nondegenerate annulus. Choose any biholomorphism $\phi : A \to A(r, R)$ to a round annulus. Define

$$\operatorname{mod}(A) := \operatorname{mod}\bigl(A(r, R)\bigr) = \frac{1}{2\pi}\log\Bigl(\frac{R}{r}\Bigr).$$

By Theorem 2.12, this number is independent of the choice of ϕ and of (r, R).

Remark 2.14 (Degenerate cases). If one complementary component collapses to a point, then A is conformally equivalent to a punctured disk (or to $\mathbb{C} \setminus \{0\}$), and the modulus is naturally taken to be $+\infty$. We focus on the nondegenerate case throughout, since it is exactly the regime where $\operatorname{mod}(A) \in (0, \infty)$ is a finite quantitative parameter; see [1, 18].

2.5. **Automorphisms of a round annulus and why modulus is the only invariant.** The next result is not strictly necessary for the definition of modulus, but it is conceptually crucial: once $\operatorname{mod}(A(r, R))$ is fixed, there are essentially no continuous conformal degrees of freedom left.

Theorem 2.15 (Automorphisms of a round annulus)**.** *Let $0 < r < R < \infty$ and let $A = A(r, R)$. Every biholomorphism $f : A \to A$ has one of the forms*

$$f(z) = e^{i\theta} z \qquad \text{or} \qquad f(z) = e^{i\theta}\,\frac{rR}{z}$$

for some real θ. In particular, the automorphism group is generated by rotations and the involution $z \mapsto rR/z$.

Proof. Let $S_{r,R} = \{w : \log r < \Re w < \log R\}$. By Proposition 2.9, $\exp : S_{r,R} \to A$ is a covering map with deck group generated by $T(w) = w + 2\pi i$. Since $S_{r,R}$ is simply connected, the holomorphic map $f \circ \exp$ admits a holomorphic lift $F : S_{r,R} \to S_{r,R}$ satisfying $\exp(F(w)) = f(\exp(w))$. From $\exp(F(w + 2\pi i)) = \exp(F(w))$ we obtain $F(w + 2\pi i) - F(w) \in 2\pi i\mathbb{Z}$ for each w, and therefore

$$k(w) := \frac{F(w + 2\pi i) - F(w)}{2\pi i}$$

is a holomorphic map from $S_{r,R}$ into $\mathbb{Z}$. Since $S_{r,R}$ is connected, $k(w)$ is constant, so $F(w+2\pi i) = F(w) + 2\pi ik$ for a fixed integer k. The induced map on $\pi_1(A) \cong \mathbb{Z}$ is multiplication by k, and since f is an automorphism, $k = \pm 1$.

The strip $S_{r,R}$ is bounded by the two vertical lines $\Re w = \log r$ and $\Re w = \log R$. A biholomorphism of $S_{r,R}$ must send boundary lines to boundary lines, so it either preserves these two boundary components or swaps them. Equivalently, the harmonic function $\Re w$ with constant boundary values on these lines is carried by F to a harmonic function with the same boundary constants, possibly exchanged. By uniqueness of the corresponding Dirichlet problem in a strip, $\Re(F(w))$ must be affine in $\Re w$ with slope $+1$ in the preserving case and slope -1 in the swapping case. The Cauchy–Riemann equations then force F itself to be affine:

$$F(w) = \alpha w + \beta \quad \text{with} \quad \alpha \in \mathbb{R} \setminus \{0\},\ \beta \in \mathbb{C}.$$

Using the period relation,

$$F(w + 2\pi i) - F(w) = \alpha(2\pi i) = 2\pi ik,$$

we obtain $\alpha = k = \pm 1$. Thus $F(w) = w + \beta$ or $F(w) = -w + \beta$. Pushing down by exponentiation gives either

$$f(e^w) = e^{w+\beta} = e^\beta e^w \quad \Rightarrow \quad f(z) = e^\beta z,$$

or

$$f(e^w) = e^{-w+\beta} = e^\beta e^{-w} \quad \Rightarrow \quad f(z) = \frac{e^\beta}{z}.$$

In the first case, mapping $A(r, R)$ to itself forces $|e^\beta| = 1$, so $e^\beta = e^{i\theta}$ and $f(z) = e^{i\theta} z$. In the second case, preserving the set of radii and swapping the boundary circles forces $|e^\beta| = rR$, so $e^\beta = rR\, e^{i\theta}$ and $f(z) = e^{i\theta} rR/z$. This is the standard automorphism classification for round annuli; see [3, 10, 4]. □

2.6. **A few immediate quantitative consequences.**

Proposition 2.16 (Uniqueness of modulus among round annuli)**.** *If $A(r, R)$ and $A(r', R')$ are biholomorphic, then*

$$\operatorname{mod}(A(r, R)) = \operatorname{mod}(A(r', R')).$$

Proof. The uniqueness statement in Theorem 2.12 already implies the conclusion. A direct proof that also reinforces the strip picture proceeds by lifting a biholomorphism to a map between strips.

Let $f : A(r, R) \to A(r', R')$ be biholomorphic and let $S_{r,R}$ and $S_{r',R'}$ be the associated strips. The map $f \circ \exp$ lifts to a holomorphic map $F : S_{r,R} \to S_{r',R'}$ with $\exp(F(w)) = f(e^w)$. As before, $F(w + 2\pi i) = F(w) + 2\pi i k$ for an integer k, and since f induces an isomorphism on $\pi_1 \cong \mathbb{Z}$, we must have $k = \pm 1$. The same boundary-line argument used in the automorphism theorem shows F is affine with real slope: $F(w) = \alpha w + \beta$ with $\alpha \in \mathbb{R} \setminus \{0\}$, and the period relation forces $\alpha = k = \pm 1$. Thus $\Re(F(w)) = \pm \Re(w) + \Re(\beta)$, and F maps the boundary lines $\Re w = \log r$ and $\Re w = \log R$ to the boundary lines $\Re w = \log r'$ and $\Re w = \log R'$ (possibly swapped), preserving the distance between them:

$$(\log R) - (\log r) = (\log R') - (\log r').$$

Equivalently, $\log(R/r) = \log(R'/r')$, and dividing by 2π gives $\operatorname{mod}(A(r, R)) = \operatorname{mod}(A(r', R'))$. □

Proposition 2.17 (Monotonicity for concentric annuli)**.** *If $0 < r_2 \le r_1 < R_1 \le R_2 < \infty$ and $A(r_1, R_1) \subset A(r_2, R_2)$, then*

$$\operatorname{mod}\big(A(r_1, R_1)\big) \le \operatorname{mod}\big(A(r_2, R_2)\big).$$

Proof. By Definition 2.4, the desired inequality is equivalent to

$$\log\Big(\frac{R_1}{r_1}\Big) \le \log\Big(\frac{R_2}{r_2}\Big).$$

Since $R_1 \le R_2$ we have $R_1/R_2 \le 1$, and since $r_2 \le r_1$ we have $r_2/r_1 \le 1$. Multiplying yields

$$\frac{R_1}{R_2} \cdot \frac{r_2}{r_1} \le 1,$$

which rearranges to $R_1/r_1 \le R_2/r_2$. Applying the increasing function log gives the claimed inequality, and dividing by 2π returns to modulus. □

Remark 2.18 (What we have accomplished in Section 2)*.* At this point we have a concrete and fully explicit dictionary for round annuli written in a way that will be reused later. The geometry can be parameterized by (r, R) or by the ratio $q = r/R \in (0, 1)$, and the modulus is $\operatorname{mod}(A(r, R)) = \frac{1}{2\pi} \log(R/r) = \frac{1}{2\pi} \log(1/q)$. The logarithmic coordinate $w = \log z$ identifies $A(r, R)$ with the strip $\log r < \Re w < \log R$ modulo $2\pi i\mathbb{Z}$, so the width $W = \log(R/r) = 2\pi \operatorname{mod}(A)$

is literally the Euclidean length of the associated cylinder model. Once $\mathrm{mod}(A)$ is fixed, the annulus has no further continuous conformal degrees of freedom, and the holomorphic automorphisms are generated by rotations and the inversion $z \mapsto rR/z$. These facts are standard and will serve as the normalization layer underneath the variational and metric estimates in the next sections; compare [1, 4, 18, 21].

3. Extremal length, capacity, and the modulus as a variational quantity

3.1. Extremal length of curve families.

Definition 3.1 (Curve family)**.** Let $\Omega \subset \mathbb{C}$ be a domain. A *curve* in Ω means a continuous map $\gamma : [a, b] \to \Omega$ which is rectifiable (so $\int_\gamma |dz| < \infty$). A *curve family* Γ in Ω is any collection of such curves.

Definition 3.2 (Borel conformal metric and its ρ-area)**.** Let $\Omega \subset \mathbb{C}$. A *Borel conformal metric* on Ω is a Borel function $\rho : \Omega \to [0, \infty]$. Its *$\rho$-area* is

$$\mathrm{Area}_\rho(\Omega) := \int_\Omega \rho(z)^2 \, dA(z),$$

where dA is planar Lebesgue area measure. We say ρ is *admissible* if

$$0 < \mathrm{Area}_\rho(\Omega) < \infty.$$

This is the standard setup for extremal length in the plane; see [12, 18, 21, 17].

Definition 3.3 (ρ-length of a curve)**.** Let ρ be a Borel conformal metric on Ω. For a rectifiable curve $\gamma : [a, b] \to \Omega$, define its *$\rho$-length* by

$$\ell_\rho(\gamma) := \int_\gamma \rho \, |dz| := \int_a^b \rho(\gamma(t)) \, |\gamma'(t)| \, dt,$$

where $\gamma'(t)$ exists almost everywhere for rectifiable curves, and the integral is understood in the standard arc-length sense.

Definition 3.4 (Extremal length)**.** Let Γ be a curve family in a domain Ω. Define the *ρ-systole* of Γ by

$$L_\rho(\Gamma) := \inf_{\gamma \in \Gamma} \ell_\rho(\gamma).$$

The *extremal length* of Γ in Ω is

$$\mathrm{Ext}_\Omega(\Gamma) := \sup_\rho \frac{L_\rho(\Gamma)^2}{\mathrm{Area}_\rho(\Omega)},$$

where the supremum ranges over admissible Borel conformal metrics ρ on Ω. This definition and its basic properties can be found in [12, 18, 21, 17].

Remark 3.5 (Scale invariance). If ρ is admissible and $c > 0$, then $c\rho$ is admissible and

$$\frac{L_{c\rho}(\Gamma)^2}{\mathrm{Area}_{c\rho}(\Omega)} = \frac{(cL_\rho(\Gamma))^2}{\int_\Omega (c\rho)^2 \, dA} = \frac{c^2 L_\rho(\Gamma)^2}{c^2 \, \mathrm{Area}_\rho(\Omega)} = \frac{L_\rho(\Gamma)^2}{\mathrm{Area}_\rho(\Omega)}.$$

Thus the ratio is invariant under scaling of ρ, so extremal length is genuinely a conformal shape quantity rather than an absolute scale quantity [12, 21].

Proposition 3.6 (Conformal invariance of extremal length)**.** *Let $\phi : \Omega \to \Omega'$ be a biholomorphism, and let Γ be a curve family in Ω. Let $\phi(\Gamma) := \{\phi \circ \gamma : \gamma \in \Gamma\}$ be the image family in Ω'. Then*

$$\mathrm{Ext}_\Omega(\Gamma) = \mathrm{Ext}_{\Omega'}(\phi(\Gamma)).$$

Proof. Let ρ' be an admissible metric on Ω' and define a metric ρ on Ω by the pullback rule

$$\rho(z) := \rho'(\phi(z))\,|\phi'(z)|.$$

For a rectifiable curve $\gamma : [a,b] \to \Omega$,

$$\ell_\rho(\gamma) = \int_a^b \rho(\gamma(t))\,|\gamma'(t)|\,dt = \int_a^b \rho'(\phi(\gamma(t)))\,|\phi'(\gamma(t))|\,|\gamma'(t)|\,dt.$$

Since ϕ is holomorphic, the chain rule holds almost everywhere along rectifiable curves: $(\phi\circ\gamma)'(t) = \phi'(\gamma(t))\,\gamma'(t)$, hence $|(\phi\circ\gamma)'(t)| = |\phi'(\gamma(t))|\,|\gamma'(t)|$ almost everywhere. Substituting yields

$$\ell_\rho(\gamma) = \int_a^b \rho'(\phi(\gamma(t)))\,|(\phi\circ\gamma)'(t)|\,dt = \ell_{\rho'}(\phi\circ\gamma).$$

Taking the infimum over $\gamma \in \Gamma$ gives $L_\rho(\Gamma) = L_{\rho'}(\phi(\Gamma))$.

For the ρ-area we compute

$$\mathrm{Area}_\rho(\Omega) = \int_\Omega \rho(z)^2\,dA(z) = \int_\Omega \rho'(\phi(z))^2\,|\phi'(z)|^2\,dA(z).$$

Because ϕ is conformal, the real Jacobian determinant satisfies $J_\phi(z) = |\phi'(z)|^2$, and change of variables gives

$$\int_\Omega \rho'(\phi(z))^2\,|\phi'(z)|^2\,dA(z) = \int_{\Omega'} \rho'(w)^2\,dA(w) = \mathrm{Area}_{\rho'}(\Omega').$$

Therefore

$$\frac{L_{\rho'}(\phi(\Gamma))^2}{\mathrm{Area}_{\rho'}(\Omega')} = \frac{L_\rho(\Gamma)^2}{\mathrm{Area}_\rho(\Omega)} \le \sup_\eta \frac{L_\eta(\Gamma)^2}{\mathrm{Area}_\eta(\Omega)} = \mathrm{Ext}_\Omega(\Gamma),$$

and taking the supremum over admissible ρ' yields $\mathrm{Ext}_{\Omega'}(\phi(\Gamma)) \le \mathrm{Ext}_\Omega(\Gamma)$. Applying the same argument to $\phi^{-1} : \Omega' \to \Omega$ and the family $\phi(\Gamma)$ gives the reverse inequality, so equality holds. This is a standard foundational fact about extremal length; see [12, 18, 21, 17]. □

3.2. **The two dual curve families on an annulus.**

Definition 3.7 (Radial and separating families)**.** Let A be a nondegenerate annulus in $\mathbb{C}$ with boundary components $\partial_0 A$ and $\partial_1 A$. The *connecting* (or *radial*) family is

$$\Gamma_{\mathrm{rad}}(A) := \{\gamma \subset A : \gamma \text{ is rectifiable and intersects both } \partial_0 A \text{ and } \partial_1 A\}.$$

The *separating* (or *circular*) family is

$$\Gamma_{\mathrm{sep}}(A) := \{\gamma \subset A : \gamma \text{ is a rectifiable closed curve in } A \text{ that separates } \partial_0 A \text{ from } \partial_1 A\}.$$

These are the two canonical "dual" families on an annulus and they are central in the extremal-length description of modulus; see [12, 18, 21].

Remark 3.8 (In the round case)*.* If $A = A(r,R)$ is round, then Γ_{rad} contains every curve joining the circles $|z| = r$ and $|z| = R$, and Γ_{sep} contains in particular every circle $|z| = \rho$ with $r < \rho < R$.

Theorem 3.9 (Extremal length equals modulus)**.** *Let A be a nondegenerate annulus. Then*

$$\mathrm{Ext}_A\big(\Gamma_{\mathrm{rad}}(A)\big) = \mathrm{mod}(A), \qquad \mathrm{Ext}_A\big(\Gamma_{\mathrm{sep}}(A)\big) = \frac{1}{\mathrm{mod}(A)},$$

and therefore

$$\mathrm{Ext}_A(\Gamma_{\mathrm{rad}}(A)) \cdot \mathrm{Ext}_A(\Gamma_{\mathrm{sep}}(A)) = 1.$$

This identification of modulus with extremal length is classical; see [12, 18, 21, 17]*.*

Proof. By conformal invariance (Proposition 3.6), it suffices to work with a round annulus. Fix $0 < r < R$ and set $A := A(r, R)$, and write

$$m := \operatorname{mod}(A) = \frac{1}{2\pi} \log\Big(\frac{R}{r}\Big).$$

We first compute $\operatorname{Ext}_A(\Gamma_{\mathrm{rad}})$ and then $\operatorname{Ext}_A(\Gamma_{\mathrm{sep}})$.

To obtain a lower bound for $\operatorname{Ext}_A(\Gamma_{\mathrm{rad}})$, introduce the candidate metric

$$\rho_*(z) := \frac{1}{|z|} \mathbf{1}_A(z),$$

so $\rho_*(z) = 1/|z|$ on A and zero outside. In polar coordinates $z = \rho e^{i\theta}$ with $dA = \rho\, d\rho\, d\theta$, its ρ-area is

$$\operatorname{Area}_{\rho_*}(A) = \int_0^{2\pi} \int_r^R \Big(\frac{1}{\rho}\Big)^2 \rho\, d\rho\, d\theta = \int_0^{2\pi} \int_r^R \frac{1}{\rho}\, d\rho\, d\theta = 2\pi \log\Big(\frac{R}{r}\Big).$$

Now let $\gamma : [a, b] \to A$ be any rectifiable curve meeting both boundary circles. Define $s(t) := |\gamma(t)|$. Since s is the composition of γ with the 1-Lipschitz map $z \mapsto |z|$, the a.e. derivative satisfies $|s'(t)| \le |\gamma'(t)|$. The ρ_*-length of γ is

$$\ell_{\rho_*}(\gamma) = \int_a^b \frac{|\gamma'(t)|}{|\gamma(t)|}\, dt = \int_a^b \frac{|\gamma'(t)|}{s(t)}\, dt \ge \int_a^b \frac{|s'(t)|}{s(t)}\, dt.$$

Since s is absolutely continuous on $[a, b]$ and takes values between r and R while hitting both endpoints, we have

$$\int_a^b \frac{|s'(t)|}{s(t)}\, dt \ge \left| \int_a^b \frac{s'(t)}{s(t)}\, dt \right| = \big| \log s(b) - \log s(a) \big|.$$

Restricting to the sub-arc from the first hit of $|z| = r$ to the last hit of $|z| = R$ ensures $s(a) = r$ and $s(b) = R$, so $\ell_{\rho_*}(\gamma) \ge \log(R/r)$ for every $\gamma \in \Gamma_{\mathrm{rad}}$ and therefore

$$L_{\rho_*}(\Gamma_{\mathrm{rad}}) \ge \log\Big(\frac{R}{r}\Big).$$

Inserting into the extremal-length quotient gives

$$\operatorname{Ext}_A(\Gamma_{\mathrm{rad}}) \ge \frac{L_{\rho_*}(\Gamma_{\mathrm{rad}})^2}{\operatorname{Area}_{\rho_*}(A)} \ge \frac{\big(\log(R/r)\big)^2}{2\pi \log(R/r)} = \frac{1}{2\pi} \log\Big(\frac{R}{r}\Big) = m.$$

For the matching upper bound, we pass to the strip model. Let $w = \log z$ so that A is the quotient of the strip $S = \{w : \log r < \Re w < \log R\}$ by $w \sim w + 2\pi i$, and fix the fundamental rectangle

$$\mathcal{R} = \{w : \log r < \Re w < \log R,\ 0 < \Im w < 2\pi\}.$$

A curve in Γ_{rad} lifts to a curve in $\mathcal{R}$ connecting the two vertical sides $\Re w = \log r$ and $\Re w = \log R$. Given any admissible metric ρ on A, define the pullback metric on $\mathcal{R}$ by

$$\widetilde{\rho}(w) := \rho(e^w)\, |e^w| = \rho(e^w)\, e^{\Re w}.$$

This is exactly $\widetilde{\rho} = \rho \circ \exp \cdot |\exp'|$, and conformal invariance shows that the extremal-length ratio computed downstairs on A equals the corresponding ratio computed upstairs on $\mathcal{R}$ for the lifted family of curves connecting the vertical sides.

For each fixed $y \in (0, 2\pi)$, consider the horizontal segment $\eta_y(x) = x + iy$ for $x \in [\log r, \log R]$. It connects the vertical sides, hence belongs to the lifted connecting family, so the lifted systole satisfies

$$L_{\widetilde{\rho}}(\widetilde{\Gamma}_{\mathrm{rad}}) \le \ell_{\widetilde{\rho}}(\eta_y) = \int_{\log r}^{\log R} \widetilde{\rho}(x + iy)\, dx$$

for every y. Squaring and applying Cauchy–Schwarz in the x-variable yields

$$L_{\widetilde{\rho}}(\widetilde{\Gamma}_{\mathrm{rad}})^2 \le \left(\int_{\log r}^{\log R} \widetilde{\rho}(x+iy)\,dx\right)^2 \le \left(\int_{\log r}^{\log R} 1\,dx\right)\left(\int_{\log r}^{\log R} \widetilde{\rho}(x+iy)^2\,dx\right).$$

The factor $\int_{\log r}^{\log R} 1\,dx$ equals $\log R - \log r = \log(R/r)$, hence

$$L_{\widetilde{\rho}}(\widetilde{\Gamma}_{\mathrm{rad}})^2 \le \log(R/r) \int_{\log r}^{\log R} \widetilde{\rho}(x+iy)^2\,dx.$$

Integrating over $y \in (0, 2\pi)$ gives

$$2\pi\, L_{\widetilde{\rho}}(\widetilde{\Gamma}_{\mathrm{rad}})^2 \le \log(R/r) \int_0^{2\pi} \int_{\log r}^{\log R} \widetilde{\rho}(x+iy)^2\,dx\,dy = \log(R/r)\ \mathrm{Area}_{\widetilde{\rho}}(\mathcal{R}),$$

so

$$\frac{L_{\widetilde{\rho}}(\widetilde{\Gamma}_{\mathrm{rad}})^2}{\mathrm{Area}_{\widetilde{\rho}}(\mathcal{R})} \le \frac{1}{2\pi}\log(R/r) = m.$$

Because this holds for every admissible ρ, taking the supremum yields $\mathrm{Ext}_A(\Gamma_{\mathrm{rad}}) \le m$. Combined with the lower bound, this proves $\mathrm{Ext}_A(\Gamma_{\mathrm{rad}}) = m$.

To compute $\mathrm{Ext}_A(\Gamma_{\mathrm{sep}})$, we again use the strip model, but now we track curves that connect the horizontal sides of $\mathcal{R}$. Let Γ_V denote the family of curves in $\mathcal{R}$ connecting the vertical sides, and let Γ_H denote the family connecting the horizontal sides. Repeating the computation above with the roles of x and y swapped shows

$$\mathrm{Ext}_{\mathcal{R}}(\Gamma_V) = \frac{\log(R/r)}{2\pi}, \qquad \mathrm{Ext}_{\mathcal{R}}(\Gamma_H) = \frac{2\pi}{\log(R/r)},$$

so $\mathrm{Ext}_{\mathcal{R}}(\Gamma_V)\mathrm{Ext}_{\mathcal{R}}(\Gamma_H) = 1$. Under the quotient identifying $\Im w = 0$ with $\Im w = 2\pi$, a curve in $\mathcal{R}$ that connects the horizontal sides projects to a closed curve in the annulus that winds once around the hole and therefore separates the boundary components. Conversely, any separating closed curve in A lifts to a path in the strip whose endpoints differ by $2\pi i$, and in the fundamental rectangle it appears as a curve connecting the horizontal sides. Thus the lift of $\Gamma_{\mathrm{sep}}(A)$ is precisely the family Γ_H in $\mathcal{R}$. Conformal invariance therefore yields

$$\mathrm{Ext}_A(\Gamma_{\mathrm{sep}}) = \mathrm{Ext}_{\mathcal{R}}(\Gamma_H) = \frac{2\pi}{\log(R/r)} = \frac{1}{m},$$

and multiplying the two values gives the reciprocity identity. This rectangle computation and its transport to annuli is a standard extremal-length argument; see [12, 18, 17, 21]. □

3.3. **Capacity of a condenser and its relation to modulus.**

Definition 3.10 (Dirichlet energy)**.** Let $\Omega \subset \mathbb{C}$ be a domain and let $u : \Omega \to \mathbb{R}$ be weakly differentiable. Define the *Dirichlet energy* of u by

$$\mathcal{E}_\Omega(u) := \int_\Omega |\nabla u|^2\,dA = \int_\Omega \left(u_x^2 + u_y^2\right)\,dA.$$

This energy and its minimization principles are standard in potential theory; see [27, 11, 28].

Definition 3.11 (Condenser capacity)**.** Let A be a nondegenerate annulus with boundary components $\partial_0 A, \partial_1 A$. Define its *capacity* by

$$\Cap(A) := \inf\left\{\mathcal{E}_A(u) \,:\, u \in C^\infty(A),\ 0 \le u \le 1,\ u|_{\partial_0 A} = 0,\ u|_{\partial_1 A} = 1\right\},$$

where boundary conditions are understood in the classical sense when ∂A is smooth, and otherwise in the standard Sobolev trace sense. This is the classical condenser capacity associated to the two boundary components; see [27, 11, 18].

Theorem 3.12 (Capacity of a round annulus and its extremizer)**.** *Let $A = A(r, R)$. Then*

$$\Cap(A) = \frac{2\pi}{\log(R/r)}.$$

Moreover, the unique energy minimizer is the harmonic function

$$u_*(z) := \frac{\log|z| - \log r}{\log R - \log r},$$

which satisfies $u_* = 0$ on $|z| = r$ and $u_* = 1$ on $|z| = R$. This computation is classical and appears in many sources; see [27, 11, 18, 12]*.*

Proof. Let $L := \log R - \log r = \log(R/r)$ and define $u_*(z) = \frac{1}{L}(\log|z| - \log r)$. On $|z| = r$ we have $\log|z| = \log r$ and therefore $u_* = 0$, and on $|z| = R$ we have $\log|z| = \log R$ and therefore $u_* = 1$. The function u_* is smooth on A since $\log|z|$ is smooth away from 0.

We compute ∇u_* explicitly. Writing $z = x + iy$,

$$u_*(z) = \frac{1}{L}\big(\log|z| - \log r\big), \qquad \nabla u_* = \frac{1}{L}\nabla(\log|z|).$$

Since $\log|z| = \frac{1}{2}\log(x^2 + y^2)$,

$$\frac{\partial}{\partial x}\log|z| = \frac{x}{x^2 + y^2}, \qquad \frac{\partial}{\partial y}\log|z| = \frac{y}{x^2 + y^2},$$

and therefore $\nabla(\log|z|) = \frac{1}{|z|^2}(x, y)$. It follows that

$$|\nabla u_*(z)|^2 = \frac{1}{L^2}\cdot\frac{x^2 + y^2}{|z|^4} = \frac{1}{L^2}\cdot\frac{1}{|z|^2}.$$

In polar coordinates $z = \rho e^{i\theta}$ with $dA = \rho\, d\rho\, d\theta$, this gives

$$\mathcal{E}_A(u_*) = \int_0^{2\pi}\int_r^R \left(\frac{1}{L^2}\cdot\frac{1}{\rho^2}\right)\rho\, d\rho\, d\theta = \frac{1}{L^2}\int_0^{2\pi}\int_r^R \frac{1}{\rho}\, d\rho\, d\theta = \frac{1}{L^2}\int_0^{2\pi} L\, d\theta = \frac{2\pi}{L}.$$

Hence $\Cap(A) \le \mathcal{E}_A(u_*) = \frac{2\pi}{\log(R/r)}$.

For the reverse inequality, let u be any admissible function with $u = 0$ on $|z| = r$ and $u = 1$ on $|z| = R$. Fix $\theta \in [0, 2\pi]$ and consider the radial segment $\rho \mapsto \rho e^{i\theta}$. Define $U_\theta(\rho) := u(\rho e^{i\theta})$. By the chain rule and the fact that $\left|\frac{d}{d\rho}(\rho e^{i\theta})\right| = 1$,

$$|U_\theta'(\rho)| \le |\nabla u(\rho e^{i\theta})|.$$

Using the boundary values, $U_\theta(r) = 0$ and $U_\theta(R) = 1$, so

$$1 = U_\theta(R) - U_\theta(r) = \int_r^R U_\theta'(\rho)\, d\rho \le \int_r^R |U_\theta'(\rho)|\, d\rho \le \int_r^R |\nabla u(\rho e^{i\theta})|\, d\rho.$$

Apply Cauchy–Schwarz in ρ with the weight that matches the polar area element:

$$\int_r^R |\nabla u(\rho e^{i\theta})|\, d\rho = \int_r^R \big(|\nabla u(\rho e^{i\theta})|\sqrt{\rho}\big)\cdot\frac{1}{\sqrt{\rho}}\, d\rho \le \left(\int_r^R |\nabla u(\rho e^{i\theta})|^2\, \rho\, d\rho\right)^{1/2}\left(\int_r^R \frac{1}{\rho}\, d\rho\right)^{1/2}.$$

Since $\int_r^R \frac{1}{\rho}\, d\rho = L$, we obtain

$$1 \le \sqrt{L}\left(\int_r^R |\nabla u(\rho e^{i\theta})|^2\, \rho\, d\rho\right)^{1/2},$$

hence

$$\int_r^R |\nabla u(\rho e^{i\theta})|^2\, \rho\, d\rho \ge \frac{1}{L}.$$

Curve family	Extremal length	Extremizer (round case)
Γ_{rad} (connect boundaries)	$\mathrm{mod}(A)$	$\rho_*(z) = 1/\lvert z \rvert$
Γ_{sep} (separate boundaries)	$1/\mathrm{mod}(A)$	rectangle dual metric upstairs

Table 1. Variational identities showing that modulus is the extremal length of the connecting family and the reciprocal extremal length of the separating family.

Integrating in θ gives

$$\int_0^{2\pi} \int_r^R |\nabla u(\rho e^{i\theta})|^2 \, \rho \, d\rho \, d\theta \geq \int_0^{2\pi} \frac{1}{L} \, d\theta = \frac{2\pi}{L}.$$

The left-hand side is exactly $\int_A |\nabla u|^2 \, dA = \mathcal{E}_A(u)$, so every admissible u satisfies $\mathcal{E}_A(u) \geq \frac{2\pi}{\log(R/r)}$. Together with the matching upper bound from u_*, this proves $⋒(A) = \frac{2\pi}{\log(R/r)}$ and that u_* is a minimizer.

Uniqueness follows from the strict convexity of the Dirichlet energy: if u_1 and u_2 are minimizers, then for $t \in (0, 1)$ the convex combination $u_t = (1-t)u_1 + tu_2$ satisfies

$$\mathcal{E}_A(u_t) \leq (1-t)\mathcal{E}_A(u_1) + t\mathcal{E}_A(u_2)$$

with equality only when $\nabla u_1 = \nabla u_2$ almost everywhere, which forces $u_1 - u_2$ to be constant. The boundary conditions force that constant to be 0, so $u_1 = u_2$. □

Corollary 3.13 (Capacity–modulus relation)**.** *For a round annulus $A(r, R)$,*

$$⋒(A(r, R)) = \frac{1}{\mathrm{mod}(A(r, R))}.$$

Equivalently,

$$\mathrm{mod}(A(r, R)) = \frac{1}{⋒(A(r, R))}.$$

Proof. By Definition 2.4 and Theorem 3.12,

$$⋒(A(r, R)) = \frac{2\pi}{\log(R/r)} = \frac{1}{\frac{1}{2\pi}\log(R/r)} = \frac{1}{\mathrm{mod}(A(r, R))}.$$

Rearranging gives the second identity. This relationship is classical and is discussed in [18, 27, 21]. □

3.4. **Geometric meaning.** We now translate the computed equalities into geometric statements. Let A be a nondegenerate annulus and write $m = \mathrm{mod}(A)$. Theorem 3.9 says the family of curves connecting the boundary components has extremal length m, so crossing is conformally expensive when m is large. The same theorem says the separating family has extremal length $1/m$, so separating is conformally expensive when m is small. The product identity $\mathrm{Ext}(\Gamma_{\mathrm{rad}})\mathrm{Ext}(\Gamma_{\mathrm{sep}}) = 1$ expresses the precise duality between the two.

For the normalized annulus $A(q, 1)$ we have $m = \frac{1}{2\pi}\log(1/q)$. When $m \to \infty$ (equivalently $q \to 0$) the annulus becomes thin near the puncture, and the equality $\mathrm{Ext}(\Gamma_{\mathrm{rad}}) = m$ forces connecting curves to have large extremal length, while $\mathrm{Ext}(\Gamma_{\mathrm{sep}}) = 1/m$ tends to 0. When $m \to 0$ (equivalently $q \to 1$) the annulus becomes thick in the sense that the two boundary circles approach one another, and then connecting curves have small extremal length while separating curves have large extremal length. This is the first fully rigorous way in which $\mathrm{mod}(A)$ controls annular geometry through variational quantities; compare [12, 18, 21, 17].

3.5. **Figures to include.**

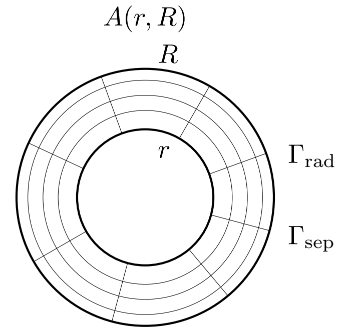


Figure 1. The dual curve families on a round annulus: connecting curves Γ_{rad} (spokes) and separating curves Γ_{sep} (loops).

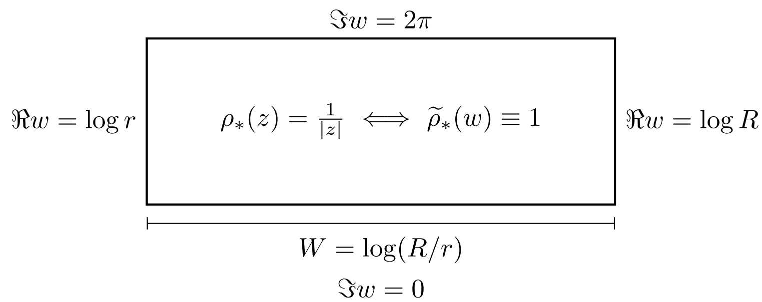


Figure 2. In logarithmic coordinates $z = e^w$, the metric $\rho_*(z) = 1/|z|$ pulls back to the constant metric on the strip.

4. Universal covering, strip-lifts, and Fourier analysis on annuli

4.1. **Covering map and fundamental domain.** We now make the "annulus $\leftrightarrow$ cylinder" dictionary operational for analysis. Throughout this section we fix a round annulus

$$A := A(r, R) = \{z \in \mathbb{C} : r < |z| < R\}, \qquad 0 < r < R < \infty,$$

and we write the logarithmic width

$$W := \log\left(\frac{R}{r}\right) = 2\pi \operatorname{mod}(A).$$

The covering picture for annuli through the exponential map is standard; see [4, 2, 3].

Definition 4.1 (Logarithmic coordinate)**.** Fix a branch of the argument on a simply connected subdomain of A. For $z \in A$ we write the logarithmic coordinate

$$w = \log z := \log |z| + i \arg z,$$

so that $z = e^w$.

Remark 4.2 (Multi-valuedness and the deck group). Because $\arg z$ is defined only modulo 2π, the logarithm is multi-valued on A. If w is one logarithm of z, then every logarithm of z is of the form $w + 2\pi i k$ with $k \in \mathbb{Z}$. This additive $\mathbb{Z}$-action is the deck action of the universal cover and is exactly the same periodicity that later drives Fourier analysis in the imaginary direction [4, 3].

Definition 4.3 (Covering strip and fundamental rectangle)**.** Define the vertical strip

$$S := S_{r,R} = \{w \in \mathbb{C} : \log r < \Re w < \log R\}.$$

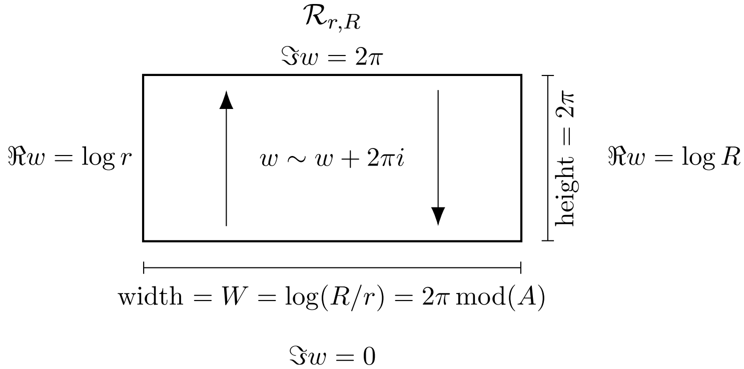


Figure 3. A fundamental rectangle for the covering strip $S_{r,R}$, with the horizontal edges identified by $w \sim w + 2\pi i$.

Define also the fundamental rectangle

$$\mathcal{R} := \mathcal{R}_{r,R} = \{w \in \mathbb{C} : \log r < \Re w < \log R,\ 0 < \Im w < 2\pi\}.$$

Proposition 4.4 (Exponential covering revisited)**.** *The exponential map* $\exp(w) = e^w$ *restricts to a holomorphic covering map*

$$\exp : S \to A.$$

Its deck group is generated by the translation $T(w) = w + 2\pi i$*. Moreover,* S *has Euclidean width* $W = \log(R/r)$ *and*

$$\frac{W}{2\pi} = \operatorname{mod}(A).$$

Proof. This is Proposition 2.9 from Section 2, rewritten with $W = \log(R/r)$ to keep the modulus scale visible. The key point is that the fiber relation $e^{w_1} = e^{w_2}$ forces $w_1 - w_2 \in 2\pi i\mathbb{Z}$, so the deck group is generated by $w \mapsto w + 2\pi i$, and the strip width is $\log R - \log r = \log(R/r)$ [4, 2, 3]. □

Remark 4.5 (Why the rectangle is the right fundamental domain)*.* The identification is

$$x + i0 \ \sim\ x + i2\pi \qquad (\log r < x < \log R),$$

because adding $2\pi i$ does not change e^w. Thus it is the horizontal edges of $\mathcal{R}$ that are identified, and the quotient $\mathcal{R}/(y \sim y + 2\pi)$ is a Euclidean cylinder of circumference 2π and length W. This is the geometric origin of the slogan "mod(A) is an aspect ratio" [1, 4].

4.2. **Holomorphic maps as periodic holomorphic functions on a strip.** The advantage of the covering picture is that holomorphic analysis on A becomes holomorphic analysis on S plus a simple periodicity constraint. This observation is classical and appears implicitly in many treatments of Laurent series and Hardy spaces on annuli [3, 7].

Definition 4.6 (Strip-lift of a holomorphic map)**.** Let $f \in \mathrm{Hol}(A, \mathbb{C})$. Define its *strip-lift* by

$$F : S \to \mathbb{C}, \qquad F(w) := f(e^w).$$

Proposition 4.7 (Holomorphicity and periodicity of the lift)**.** *Let* $f \in \mathrm{Hol}(A, \mathbb{C})$ *and let* $F(w) = f(e^w)$ *be its strip-lift. Then* F *is holomorphic on* S *and satisfies* $2\pi i$*-periodicity:*

$$F(w + 2\pi i) = F(w) \qquad \text{for all } w \in S.$$

Conversely, if F *is holomorphic and* $2\pi i$*-periodic on* S*, then there exists a unique function* $f \in \mathrm{Hol}(A, \mathbb{C})$ *such that* $F(w) = f(e^w)$ *for all* $w \in S$*.*

Proof. The map $\exp : S \to A$ is holomorphic, so $F = f \circ \exp$ is holomorphic. Periodicity follows from the identity $e^{w+2\pi i} = e^w e^{2\pi i} = e^w$, which gives

$$F(w + 2\pi i) = f(e^{w+2\pi i}) = f(e^w) = F(w).$$

For the converse, assume F is holomorphic and $2\pi i$-periodic on S. For $z \in A$, choose any $w \in S$ with $e^w = z$ and define $f(z) := F(w)$. If $w_1, w_2 \in S$ satisfy $e^{w_1} = e^{w_2}$, then $w_2 = w_1 + 2\pi i k$ for some $k \in \mathbb{Z}$, and periodicity gives $F(w_2) = F(w_1)$, so f is well-defined. To see that f is holomorphic, fix $z_0 \in A$ and choose $w_0 \in S$ with $e^{w_0} = z_0$. Since exp is locally biholomorphic at w_0, there exists a neighborhood $U \subset S$ on which exp is biholomorphic onto a neighborhood $V \subset A$ of z_0. On V we can write $f(z) = F((\exp|_U)^{-1}(z))$, a composition of holomorphic maps, hence holomorphic. Uniqueness is immediate because every $z \in A$ has some logarithm $w \in S$ and then $f(z)$ is forced to be $F(w)$. This "descend from a periodic lift" argument is a standard covering-space technique [4, 3]. □

Proposition 4.8 (Derivative conversion formula)**.** *Let $f \in \mathrm{Hol}(A, \mathbb{C})$ and $F(w) = f(e^w)$. Then for all $w \in S$,*

$$F'(w) = e^w f'(e^w).$$

Equivalently, for $z = e^w$,

$$f'(z) = \frac{1}{z} F'(\log z),$$

where $\log z$ denotes any logarithm in S.

Proof. Differentiate the identity $F(w) = f(e^w)$ and apply the chain rule:

$$F'(w) = f'(e^w) \cdot (e^w)' = f'(e^w) \cdot e^w.$$

Writing $z = e^w$ gives $F'(w) = zf'(z)$ and hence $f'(z) = F'(w)/z$. The right-hand side does not depend on the chosen logarithm because F' inherits $2\pi i$-periodicity from F [3, 7]. □

4.3. **Laurent series vs Fourier series.** The analytic payoff of the strip model is that periodicity in $\Im w$ produces Fourier series, and those Fourier modes become Laurent monomials in $z = e^w$. This correspondence is the clean conceptual bridge between classical Laurent theory on annuli and harmonic/Fourier analysis on cylinders [3, 7, 1].

Theorem 4.9 (Fourier–Laurent correspondence)**.** *Let $f \in \mathrm{Hol}(A(r, R), \mathbb{C})$ and let $F(w) = f(e^w)$ be its $2\pi i$-periodic strip-lift. Then f admits a Laurent expansion*

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

converging absolutely and uniformly on compact subsets of $A(r, R)$, and F admits a Fourier-type expansion

$$F(w) = \sum_{n \in \mathbb{Z}} a_n e^{nw}$$

converging absolutely and uniformly on compact subsets of $S_{r,R}$. The coefficient a_n can be recovered by either of the equivalent formulas

$$a_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{n+1}}\, dz \qquad (r < \rho < R),$$

and

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} F(x + iy)\, e^{-iny}\, dy \qquad (\log r < x < \log R),$$

where the second formula uses the 2π-periodic variable $y = \Im w$.

Proof. Fix ρ with $r < \rho < R$ and define, for each $n \in \mathbb{Z}$,

$$a_n := \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{n+1}}\, dz.$$

If $\rho_1 < \rho_2$ are two radii in (r, R), then the difference of these integrals equals the integral of $f(z)/z^{n+1}$ around the boundary of the annulus $\{\rho_1 < |z| < \rho_2\}$, which is zero by Cauchy's theorem since $f(z)/z^{n+1}$ is holomorphic there. Thus a_n is independent of ρ. Classical Laurent theory then shows that on any compact subannulus $A(\rho_1, \rho_2)$ with $r < \rho_1 < \rho_2 < R$, the series $\sum_{n\in\mathbb{Z}} a_n z^n$ converges uniformly to $f(z)$, and therefore converges absolutely and locally uniformly on $A(r, R)$ [3, 7].

Setting $z = e^w$ gives $z^n = e^{nw}$, hence

$$F(w) = f(e^w) = \sum_{n\in\mathbb{Z}} a_n (e^w)^n = \sum_{n\in\mathbb{Z}} a_n e^{nw},$$

with locally uniform convergence on S because compact subsets of S map under exp to compact subannuli of A.

For the Fourier extraction formula, fix x with $\log r < x < \log R$. The function $y \mapsto F(x + iy)$ is 2π-periodic and smooth, so it has Fourier coefficients. From the expansion above,

$$F(x + iy) = \sum_{n\in\mathbb{Z}} a_n e^{n(x+iy)} = \sum_{n\in\mathbb{Z}} (a_n e^{nx}) e^{iny},$$

so the nth Fourier coefficient equals $a_n e^{nx}$, which yields

$$a_n e^{nx} = \frac{1}{2\pi} \int_0^{2\pi} F(x + iy)\, e^{-iny}\, dy, \qquad a_n = \frac{1}{2\pi} \int_0^{2\pi} F(x + iy)\, e^{-iny}\, dy.$$

To match the Cauchy integral formula explicitly, parametrize $|z| = \rho$ by $z = \rho e^{iy}$ so that $dz = i\rho e^{iy} dy$. Then

$$\frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{n+1}}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{iy})\, \rho^{-n} e^{-iny}\, dy.$$

Writing $\rho = e^x$ and $f(\rho e^{iy}) = F(x + iy)$ gives the same coefficient formula upstairs. This is the standard identification of Laurent coefficients with Fourier coefficients after logarithmic lifting [3, 7, 1]. □

4.4. **Quantitative coefficient control via strip width.** We now make the dependence on the width $W = \log(R/r) = 2\pi \mathrm{mod}(A)$ explicit. The guiding principle is that holomorphy on a strip of width W forces exponential decay of Fourier modes in the periodic direction, and this becomes exponential decay of Laurent coefficients in the annulus variable [12, 21].

Definition 4.10 (Centered strip)**.** It is often convenient to recenter S by translating in the real direction. Let

$$x_0 := \frac{\log r + \log R}{2} = \log \sqrt{rR}.$$

Define the centered strip

$$S^{\mathrm{ctr}} := \{w \in \mathbb{C} : |\Re w - x_0| < W/2\}.$$

Translation $w \mapsto w - x_0$ identifies this with the symmetric strip $\{|\Re w| < W/2\}$.

Theorem 4.11 (Exponential decay of Fourier/Laurent coefficients)**.** *Let $A = A(r, R)$ and set $W = \log(R/r)$. Let $f \in \mathrm{Hol}(A, \mathbb{C})$ and write its Laurent expansion*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n.$$

Assume f is bounded on A:

$$\|f\|_{L^\infty(A)} := \sup_{z\in A} |f(z)| \le M.$$

Then for every integer $n \in \mathbb{Z}$,

$$|a_n| \le M\, e^{-|n|W/2}\, (\sqrt{rR})^{-|n|}.$$

In particular, in the normalized annulus $A(q,1)$ (so $R=1$ and $r=q$),

$$|a_n| \le M\, q^{|n|/2}.$$

Proof. Let $F(w) = f(e^w)$ be the $2\pi i$-periodic lift of f to the strip S. Then F is holomorphic and satisfies $|F(w)| \le M$ for all $w \in S$.

Fix $n \in \mathbb{Z}$ and consider the holomorphic function on S

$$G_n(w) := F(w)e^{-nw}.$$

The periodicity of F implies $G_n(w+2\pi i) = G_n(w)$ because $e^{-n(w+2\pi i)} = e^{-nw}e^{-2\pi i n} = e^{-nw}$ for integer n. For any real x with $\log r < x < \log R$, define

$$I_n(x) := \frac{1}{2\pi}\int_0^{2\pi} G_n(x+iy)\,dy = \frac{1}{2\pi}\int_0^{2\pi} F(x+iy)e^{-n(x+iy)}\,dy.$$

By Theorem 4.9, this quantity equals the Laurent coefficient a_n for every such x.

To see that $I_n(x)$ is independent of x, take $x_1 < x_2$ in $(\log r, \log R)$ and integrate G_n around the boundary of the rectangle with vertical sides $\Re w = x_1$ and $\Re w = x_2$ and horizontal sides $\Im w = 0$ and $\Im w = 2\pi$. Cauchy's theorem gives zero. The integrals on the top and bottom edges cancel because G_n is $2\pi i$-periodic and the orientations are opposite, leaving equality of the integrals along the two vertical sides. Dividing by 2π gives $I_n(x_1) = I_n(x_2)$, so $I_n(x)$ is constant.

Since $a_n = I_n(x)$ for every x, we may evaluate it on whichever vertical line gives the best bound. If $n \ge 0$, choose x arbitrarily close to $\log R$ and estimate

$$|a_n| = |I_n(x)| \le \frac{1}{2\pi}\int_0^{2\pi} |F(x+iy)|\,|e^{-n(x+iy)}|\,dy \le \frac{1}{2\pi}\int_0^{2\pi} M\,e^{-nx}\,dy = Me^{-nx}.$$

Letting $x \uparrow \log R$ yields $|a_n| \le Me^{-n\log R} = MR^{-n}$. If $n = -k \le 0$ with $k \ge 0$, choose x arbitrarily close to $\log r$ and obtain

$$|a_{-k}| \le Me^{k\log r} = Mr^k.$$

We now rewrite these two bounds so the strip width W appears symmetrically. Write $\log R = x_0 + W/2$ and $\log r = x_0 - W/2$. For $n \ge 0$,

$$R^{-n} = e^{-n\log R} = e^{-n(x_0+W/2)} = e^{-nx_0}\,e^{-nW/2}.$$

For $n = -k \le 0$,

$$r^k = e^{k\log r} = e^{k(x_0-W/2)} = e^{kx_0}\,e^{-kW/2}.$$

Since $|n| = k$ and $e^{kx_0} = (\sqrt{rR})^k = (\sqrt{rR})^{-|n|}$ is the same symmetric base factor, both cases can be written as

$$|a_n| \le M\,e^{-|n|W/2}\,(\sqrt{rR})^{-|n|}.$$

In the normalized case $R=1$, $r=q$, we have $\sqrt{rR} = \sqrt{q}$ and $W = \log(1/q)$, so $e^{-|n|W/2} = (e^{-\log(1/q)})^{|n|/2} = q^{|n|/2}$ and the formula becomes $|a_n| \le Mq^{|n|/2}$. ∎

Remark 4.12 (What the estimate is really saying). In the normalized annulus $A(q,1)$, $q = e^{-2\pi \operatorname{mod}(A)}$. Thus $q^{|n|/2} = \exp(-\pi|n|\operatorname{mod}(A))$, so every nonconstant Laurent/Fourier mode is exponentially suppressed as the modulus grows. This is the mechanism behind thin-annulus rigidity statements in later sections, and it is one of the cleanest quantitative places where the modulus enters analysis [12, 21].

Corollary 4.13 (A mid-circle oscillation bound)**.** *Let $A = A(q,1)$ and suppose $|f| \le M$ on A. Write $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$. On the core circle $|z| = \sqrt{q}$ one has*

$$\sup_{|z|=\sqrt{q}} |f(z) - a_0| \le \sum_{n\neq 0} |a_n|\,|z|^{|n|} \le 2M \sum_{n\ge 1} q^{n/2}\, q^{n/2} = 2M \sum_{n\ge 1} q^n = \frac{2Mq}{1-q}.$$

Proof. On $|z| = \sqrt{q}$, the Laurent series gives $f(z) - a_0 = \sum_{n\neq 0} a_n z^n$. Taking absolute values and using $|z|^{|n|} = q^{|n|/2}$ gives

$$|f(z) - a_0| \le \sum_{n\neq 0} |a_n|\, q^{|n|/2}.$$

The coefficient bound from Theorem 4.11 in the normalized case yields $|a_n| \le Mq^{|n|/2}$, hence each summand is bounded by $Mq^{|n|}$, and the geometric-series computation gives the claim. □

4.5. **Hardy spaces and boundary traces.** We include a minimal Hardy-space toolkit because it quantifies how boundary data on one circle controls boundary data on the other, with constants expressed in the logarithmic coordinate. The analytic core here is the convexity in $\log\rho$ of radial L^p-means, which is a form of the classical three-circles/three-lines philosophy [3, 7].

Definition 4.14 (Hardy spaces on a round annulus)**.** Let $1 \le p < \infty$ and let $f \in \mathrm{Hol}(A(r,R),\mathbb{C})$. We say $f \in H^p(A(r,R))$ if

$$\|f\|^p_{H^p(A(r,R))} := \sup_{r<\rho<R} \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^p\, d\theta < \infty.$$

Proposition 4.15 (Log-convexity in the radial variable)**.** *Let $f \in \mathrm{Hol}(A(r,R),\mathbb{C})$ and $1 \le p < \infty$. Define*

$$M_p(\rho) := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^p\, d\theta\right)^{1/p}.$$

Then the function $t \mapsto \log M_p(e^t)$ is convex on $(\log r, \log R)$.

Proof. The function $|f|^p$ is subharmonic for $p \ge 1$, so its circular means in the variable θ interact well with harmonic measure on the strip after lifting via $w = \log z$. Equivalently, the three-lines theorem applied to the holomorphic function $f(e^w)$ in the strip gives convexity in $\Re w$, which translates to convexity of $t \mapsto \log M_p(e^t)$ in t. This is a standard consequence of subharmonicity and Poisson representation in an annulus/strip and appears in many complex analysis texts in the guise of the three-circles theorem and its L^p variants [3, 7, 1]. □

Corollary 4.16 (Two-circle interpolation with modulus)**.** *Let $f \in \mathrm{Hol}(A(r,R))$ and $1 \le p < \infty$. For $\rho \in (r,R)$ write*

$$\alpha := \frac{\log(R/\rho)}{\log(R/r)} \in (0,1), \qquad 1-\alpha = \frac{\log(\rho/r)}{\log(R/r)}.$$

Then

$$M_p(\rho) \le M_p(r)^{\alpha}\, M_p(R)^{1-\alpha}.$$

The weights are affine in the logarithmic coordinate and depend only on the normalized position of ρ inside the strip of width $W = 2\pi\mathrm{mod}(A)$.

Proof. Convexity of $t \mapsto \log M_p(e^t)$ implies that for $t = \log\rho$,

$$\log M_p(\rho) \le \alpha \log M_p(r) + (1-\alpha)\log M_p(R),$$

where $\alpha = \frac{\log R - \log\rho}{\log R - \log r}$. Exponentiating yields the claim. □

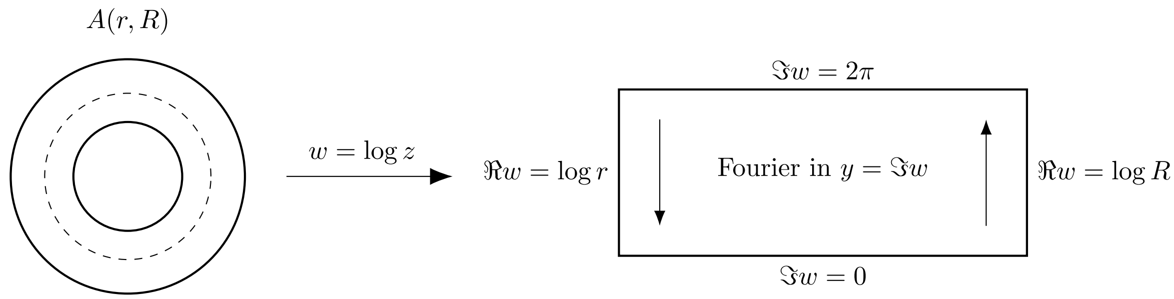


Figure 4. The logarithmic change of variables sends the annulus to a strip fundamental rectangle. Periodicity in $\Im w$ produces Fourier series in y, which become Laurent series after $z = e^w$.

4.6. **Figures to include.** The fundamental rectangle picture has already been introduced in Figure 3, right where the quotient identification is defined, since it is most helpful there. We also record a second small visual template that emphasizes the "annulus → strip" map and where Fourier modes live.

5. Intrinsic metrics on annuli and explicit dependence on modulus

5.1. **Hyperbolic metric on doubly connected domains.** We now put a canonical intrinsic metric on an annulus A, and we make its dependence on

$$\mathrm{mod}(A) = \frac{1}{2\pi} \log\left(\frac{R}{r}\right)$$

completely explicit in the round model $A(r,R)$. This will be the geometric engine behind Schwarz–Pick type estimates later. Background on hyperbolic metrics on planar domains and their conformal naturality can be found in [12, 33, 30, 34].

Definition 5.1 (Conformal Riemannian metric and its density)**.** Let $\Omega \subset \mathbb{C}$. A conformal (Riemannian) metric on Ω is a line element of the form

$$ds_\Omega = \rho_\Omega(z)\,|dz|,$$

where $\rho_\Omega : \Omega \to (0,\infty)$ is a positive function (typically C^2). The function ρ_Ω is called the metric density.

Definition 5.2 (Gaussian curvature in conformal coordinates)**.** Let $ds = \rho(z)|dz|$ be a C^2 conformal metric on Ω. Its Gaussian curvature is

$$K(z) = -\frac{\Delta(\log \rho)(z)}{\rho(z)^2},$$

where $\Delta = \partial_{xx} + \partial_{yy}$ is the Euclidean Laplacian.

Remark 5.3 (Why this is the correct formula). For a conformal metric $ds^2 = \rho^2(dx^2 + dy^2)$, the Christoffel symbols simplify and the curvature collapses to the scalar expression in Definition 5.2. We will use this formula as a computational tool below and as a mnemonic for how curvature reacts to conformal pullbacks. Standard derivations can be found in treatments of two-dimensional conformal geometry and hyperbolic metrics [30, 34].

Definition 5.4 (The hyperbolic metric on the upper half-plane)**.** On the upper half-plane

$$\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\},$$

define the line element

$$ds_{\mathbb{H}} = \rho_{\mathbb{H}}(z)\,|dz|, \qquad \rho_{\mathbb{H}}(z) := \frac{1}{\Im z} = \frac{1}{y}.$$

Proposition 5.5 (Curvature and completeness of $(\mathbb{H}, ds_{\mathbb{H}})$)**.** *The metric $ds_{\mathbb{H}}$ has constant curvature $K \equiv -1$ and is complete.*

Proof. Here $\rho_{\mathbb{H}}(x+iy) = 1/y$, so $\log \rho_{\mathbb{H}} = -\log y$. Differentiating gives $\partial_x(\log \rho_{\mathbb{H}}) = 0$ and $\partial_{xx}(\log \rho_{\mathbb{H}}) = 0$, while $\partial_y(\log \rho_{\mathbb{H}}) = -1/y$ and $\partial_{yy}(\log \rho_{\mathbb{H}}) = 1/y^2$. Hence $\Delta(\log \rho_{\mathbb{H}}) = 1/y^2$ and Definition 5.2 yields

$$K = -\frac{\Delta(\log \rho_{\mathbb{H}})}{\rho_{\mathbb{H}}^2} = -\frac{(1/y^2)}{(1/y)^2} = -1.$$

To see completeness, consider a rectifiable curve $\gamma(t) = x(t) + iy(t)$ that leaves every compact subset of $\mathbb{H}$. Either $y(t)$ approaches 0 along a sequence, or $|x(t) + iy(t)|$ tends to infinity along a sequence. If $y(t) \downarrow 0$ along some segment, then the hyperbolic length satisfies

$$\ell_{\mathbb{H}}(\gamma) = \int \frac{|\gamma'(t)|}{y(t)}\,dt \geq \int \frac{|y'(t)|}{y(t)}\,dt \geq \left| \int \frac{y'(t)}{y(t)}\,dt \right| = |\log y| \to \infty.$$

If instead $|z| \to \infty$ while y stays bounded below by $y_0 > 0$ on a tail, then $\rho_{\mathbb{H}}(z) \leq 1/y_0$ is bounded, but any path to infinity has infinite Euclidean length, hence also infinite hyperbolic length. If y does not stay bounded below, we fall back to the previous case along a subsequence. Thus every path escaping compacta has infinite length, so the metric is complete. This is standard for the half-plane model [30, 34]. □

Definition 5.6 (Hyperbolic domain and hyperbolic metric)**.** A domain $\Omega \subset \mathbb{C}$ is called hyperbolic if its complement in the Riemann sphere $\widehat{\mathbb{C}} \setminus \Omega$ contains at least two points. In this case, Ω carries a unique complete conformal metric of curvature -1, called the hyperbolic metric, written $ds_\Omega = \rho_\Omega(z)|dz|$.

Remark 5.7 (Annuli are hyperbolic)*.* A nondegenerate annulus A has complement with two nontrivial components, hence at least two points. Therefore every nondegenerate annulus is hyperbolic.

In this paper we do not need the existence/uniqueness theorem in maximal generality, because for round annuli we can write $\rho_{A(r,R)}$ explicitly from the strip model. The computation below is classical and appears in various equivalent forms in the literature on hyperbolic metrics of planar domains [33, 30, 34].

Theorem 5.8 (Explicit hyperbolic density on a round annulus)**.** *Let $A = A(r, R)$ and set*

$$W := \log\Big(\frac{R}{r}\Big) = 2\pi \operatorname{mod}(A).$$

Then the hyperbolic metric density ρ_A is radial and given explicitly by

$$\rho_A(z) = \frac{\pi}{W} \cdot \frac{1}{|z|} \cdot \csc\Big(\frac{\pi}{W} \log\Big(\frac{|z|}{r}\Big)\Big), \qquad r < |z| < R.$$

Equivalently, writing $t = \log(|z|/r) \in (0, W)$,

$$\rho_A(z) = \frac{\pi}{W|z|} \cdot \csc\Big(\frac{\pi t}{W}\Big).$$

Proof. Let $S = \{w \in \mathbb{C} : \log r < \Re w < \log R\}$ be the covering strip for A and write $W = \log R - \log r$. Translate by $\log r$ to the width-W strip $S_W = \{u \in \mathbb{C} : 0 < \Re u < W\}$ via $u = w - \log r$. Consider the holomorphic map

$$\Phi : S_W \to \mathbb{H}, \qquad \Phi(u) = \exp\Big(\frac{i\pi}{W}u\Big).$$

Writing $u = x + iy$ with $0 < x < W$ gives

$$\Phi(u) = \exp\Big(\frac{i\pi}{W}(x+iy)\Big) = \exp\Big(-\frac{\pi}{W}y\Big)\exp\Big(i\frac{\pi}{W}x\Big),$$

so $\arg\Phi(u) = \frac{\pi}{W}x \in (0,\pi)$ and hence $\Phi(u) \in \mathbb{H}$.

Pull back the hyperbolic metric from $\mathbb{H}$ via the conformal rule $\rho_{S_W}(u) = \rho_{\mathbb{H}}(\Phi(u))\,|\Phi'(u)|$. Since $\Phi'(u) = \frac{i\pi}{W}\Phi(u)$, we have $|\Phi'(u)| = \frac{\pi}{W}|\Phi(u)|$. From the explicit expression above, $|\Phi(u)| = \exp(-\frac{\pi}{W}y)$ and $\Im\Phi(u) = \exp(-\frac{\pi}{W}y)\sin(\frac{\pi}{W}x)$. Therefore

$$\rho_{\mathbb{H}}(\Phi(u)) = \frac{1}{\Im\Phi(u)} = \frac{1}{\exp(-\frac{\pi}{W}y)\sin(\frac{\pi}{W}x)},$$

and multiplying by $|\Phi'(u)|$ yields

$$\rho_{S_W}(u) = \frac{1}{\exp(-\frac{\pi}{W}y)\sin(\frac{\pi}{W}x)} \cdot \frac{\pi}{W}\exp\Big(-\frac{\pi}{W}y\Big) = \frac{\pi}{W}\cdot\frac{1}{\sin(\frac{\pi}{W}x)}.$$

Thus

$$\rho_{S_W}(x+iy) = \frac{\pi}{W}\csc\Big(\frac{\pi}{W}x\Big),$$

which depends only on $x = \Re u$.

Undoing the translation $u = w - \log r$ gives the density on S:

$$\rho_S(w) = \frac{\pi}{W}\csc\Big(\frac{\pi}{W}(\Re w - \log r)\Big).$$

Now descend to the annulus through $z = e^w$. The hyperbolic line element satisfies $\rho_S(w)|dw| = \rho_A(e^w)|d(e^w)|$, and since $|d(e^w)| = |e^w||dw| = |z||dw|$, we obtain $\rho_A(z) = \rho_S(w)/|z|$. Using $\Re w = \log|z|$ gives

$$\rho_A(z) = \frac{1}{|z|}\cdot\frac{\pi}{W}\csc\Big(\frac{\pi}{W}\big(\log|z| - \log r\big)\Big) = \frac{\pi}{W}\cdot\frac{1}{|z|}\cdot\csc\Big(\frac{\pi}{W}\log\Big(\frac{|z|}{r}\Big)\Big),$$

as claimed. Equivalent formulas appear throughout the literature on hyperbolic metrics on annuli and strips [33, 30, 34]. $\square$

Remark 5.9 (Completeness becomes transparent). From the explicit formula, as $|z| \downarrow r$ we have $\log(|z|/r) \downarrow 0$ and $\csc(\frac{\pi}{W}\log(|z|/r)) \sim \frac{W}{\pi}\cdot\frac{1}{\log(|z|/r)} \to \infty$, so the inner boundary is at infinite hyperbolic distance. The same holds as $|z| \uparrow R$. Thus $(A, \rho_A|dz|)$ is complete, in agreement with the general existence theory [33, 30].

5.2. **Comparisons: Euclidean vs hyperbolic vs quasihyperbolic metrics.** We now compare the hyperbolic density to a quasihyperbolic density built from a boundary distance. For general domains, quasihyperbolic geometry is a robust substitute for hyperbolic geometry and provides quick, explicit upper and lower bounds [23, 22]. For round annuli, the strip model suggests a particularly well-adapted "cylindrical" boundary distance.

Definition 5.10 (Hyperbolic length and distance)**.** Let Ω be hyperbolic with density ρ_Ω. For a rectifiable curve $\gamma \subset \Omega$, define

$$\ell_\Omega(\gamma) := \int_\gamma \rho_\Omega(z)\,|dz|.$$

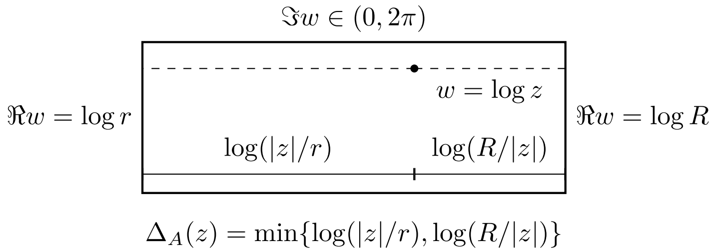


Figure 5. The cylindrical boundary distance $\Delta_A(z)$ is the distance from $\Re(\log z)$ to the boundary lines of the strip. This is the natural boundary-distance scale in which hyperbolic estimates on annuli become clean.

Define the induced distance by

$$d_\Omega(z_1, z_2) := \inf_\gamma \ell_\Omega(\gamma),$$

where the infimum runs over rectifiable curves γ in Ω joining z_1 to z_2.

Definition 5.11 (Euclidean boundary distance and quasihyperbolic density)**.** Let $\Omega \subset \mathbb{C}$ be a domain. Define the Euclidean distance to the boundary

$$\delta_\Omega(z) := \operatorname{dist}(z, \partial\Omega).$$

The quasihyperbolic density is

$$\kappa_\Omega(z) := \frac{1}{\delta_\Omega(z)},$$

and the associated quasihyperbolic length of a curve is $\int_\gamma \kappa_\Omega\,|dz|$.

Definition 5.12 (Cylindrical boundary distance on $A(r,R)$)**.** Let $A = A(r,R)$ and set $W = \log(R/r)$. For $z \in A$, define the dimensionless quantity

$$\Delta_A(z) := \min\left\{\log\Big(\frac{|z|}{r}\Big),\ \log\Big(\frac{R}{|z|}\Big)\right\}.$$

Equivalently, if $w = \log z$ with $\Re w \in (\log r, \log R)$, then $\Delta_A(z)$ is exactly the Euclidean distance from $\Re w$ to the boundary lines of the covering strip. Define the corresponding cylindrical quasihyperbolic density by

$$\kappa_A^{\mathrm{cyl}}(z) := \frac{1}{|z|\Delta_A(z)}.$$

Proposition 5.13 (Sharp comparison: hyperbolic vs cylindrical quasihyperbolic)**.** *Let $A = A(r,R)$ and let ρ_A be its hyperbolic density. Then for every $z \in A$,*

$$\kappa_A^{\mathrm{cyl}}(z) \ \le\ \rho_A(z) \ \le\ \frac{\pi}{2}\,\kappa_A^{\mathrm{cyl}}(z).$$

Equivalently,

$$\frac{1}{|z|\Delta_A(z)} \ \le\ \rho_A(z) \ \le\ \frac{\pi}{2}\cdot\frac{1}{|z|\Delta_A(z)}.$$

Proof. Write $t = \log(|z|/r) \in (0, W)$, so $W - t = \log(R/|z|)$ and $\Delta_A(z) = \min\{t, W-t\}$. By Theorem 5.8,

$$\rho_A(z) = \frac{\pi}{W|z|}\csc\Big(\frac{\pi t}{W}\Big).$$

Set $x = \frac{\pi t}{W} \in (0, \pi)$, so

$$\min\{x, \pi - x\} = \frac{\pi}{W} \min\{t, W - t\} = \frac{\pi}{W} \Delta_A(z).$$

For $x \in (0, \pi)$ one has the elementary bounds

$$\sin x \le \min\{x, \pi - x\}, \qquad \sin x \ge \frac{2}{\pi} \min\{x, \pi - x\}.$$

The first inequality follows from $\sin x \le x$ on $(0, \pi/2]$ and symmetry $\sin x = \sin(\pi - x)$ on $(\pi/2, \pi)$. The second follows from concavity of sin on $[0, \pi]$, which implies $\sin x$ lies above the chord from $(0, 0)$ to $(\pi/2, 1)$ and symmetry again, giving $\sin x \ge \frac{2}{\pi} \min\{x, \pi - x\}$. These are standard trigonometric estimates.

Using $\sin x \le \min\{x, \pi - x\} = \frac{\pi}{W} \Delta_A(z)$ gives

$$\csc x \ge \frac{W}{\pi} \cdot \frac{1}{\Delta_A(z)},$$

hence

$$\rho_A(z) = \frac{\pi}{W|z|} \csc x \ge \frac{\pi}{W|z|} \cdot \frac{W}{\pi} \cdot \frac{1}{\Delta_A(z)} = \frac{1}{|z|\Delta_A(z)} = \kappa_A^{\mathrm{cyl}}(z).$$

Using $\sin x \ge \frac{2}{\pi} \min\{x, \pi - x\} = \frac{2}{W} \Delta_A(z)$ gives

$$\csc x \le \frac{W}{2} \cdot \frac{1}{\Delta_A(z)},$$

hence

$$\rho_A(z) \le \frac{\pi}{W|z|} \cdot \frac{W}{2} \cdot \frac{1}{\Delta_A(z)} = \frac{\pi}{2} \cdot \frac{1}{|z|\Delta_A(z)} = \frac{\pi}{2} \kappa_A^{\mathrm{cyl}}(z).$$

This proves the two-sided bound. Comparable hyperbolic–quasihyperbolic inequalities on planar domains are discussed in [23, 22], and the annulus case admits this sharp trigonometric form. □

Remark 5.14 (Relation to Euclidean boundary distance and modulus). The quantity $\Delta_A(z)$ is a logarithmic boundary distance in the radial direction, while $\delta_A(z) = \mathrm{dist}(z, \partial A)$ is Euclidean. Near either boundary circle one has the local approximation $\log(1 + s) \sim s$, which yields $\Delta_A(z) \asymp \delta_A(z)/|z|$. Global comparisons between Δ_A and $\delta_A/|z|$ depend on the ratio $R/r = e^W = e^{2\pi \mathrm{mod}(A)}$ and are therefore governed by the modulus scale. In practice, Proposition 5.13 is the cleanest quantitative statement because it isolates the trigonometric factor that records the full dependence on $W = 2\pi \mathrm{mod}(A)$.

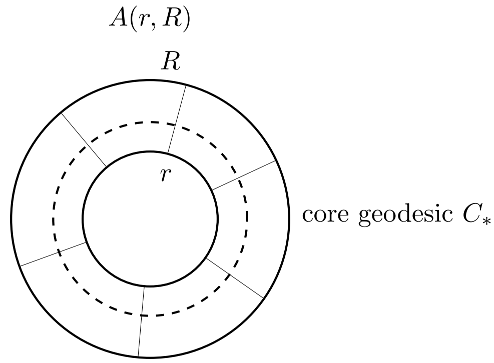
5.3. **Geodesics, collars, and the core curve.** A striking feature of annuli is that the hyperbolic metric produces a single distinguished closed geodesic in the homotopy class of circles. Its length is an explicit function of $\mathrm{mod}(A)$, and this gives an intrinsic modulus-controlled geometric invariant. Connections between hyperbolic cylinders, core geodesics, and conformal moduli are standard in hyperbolic geometry and Teichmüller theory [34, 16, 31].

Definition 5.15 (Core circle)**.** For $A(r, R)$ define the core circle

$$C_* := \{z \in \mathbb{C} : |z| = \sqrt{rR}\}.$$

Theorem 5.16 (Core geodesic and its length)**.** *Let $A = A(r, R)$ with modulus $m = \mathrm{mod}(A)$ and width $W = \log(R/r) = 2\pi m$. Then the curve C_* is a simple closed hyperbolic geodesic in A. Every simple closed curve in A homotopic to a boundary circle has hyperbolic length at least $\ell_A(C_*)$. The hyperbolic length of the core geodesic is*

$$\ell_A(C_*) = \frac{2\pi^2}{W} = \frac{\pi}{m} = \frac{\pi}{\mathrm{mod}(A)}.$$



$$\ell_A(C_*) = \frac{2\pi^2}{\log(R/r)} = \frac{\pi}{\operatorname{mod}(A)}$$

Figure 6. In the hyperbolic metric on $A(r,R)$ there is a distinguished simple closed geodesic (the core curve) at $|z| = \sqrt{rR}$, whose length is an explicit decreasing function of $\operatorname{mod}(A)$.

Proof. Use the logarithmic coordinate $z = e^w$ with $w \in S$ and write $a = \log r$, $b = \log R$, so $W = b - a$. The core circle $|z| = \sqrt{rR}$ is the image of the vertical line $\Re w = (a+b)/2 = a + W/2$, because $|z| = e^{\Re w}$.

In the proof of Theorem 5.8 we constructed the conformal map

$$\Phi(u) = \exp\Big(\frac{i\pi}{W}u\Big)$$

from the strip $S_W = \{0 < \Re u < W\}$ to $\mathbb{H}$, and this map was chosen so that pulling back the half-plane metric produces the strip hyperbolic metric. Consider the vertical line $\Re u = W/2$. For $u = W/2 + iy$,

$$\Phi\Big(\frac{W}{2} + iy\Big) = \exp\Big(\frac{i\pi}{W}\Big(\frac{W}{2} + iy\Big)\Big) = \exp\Big(i\frac{\pi}{2}\Big)\exp\Big(-\frac{\pi}{W}y\Big) = i\exp\Big(-\frac{\pi}{W}y\Big),$$

which lies on the imaginary axis in $\mathbb{H}$. The imaginary axis is a hyperbolic geodesic in $\mathbb{H}$, and because Φ is a local hyperbolic isometry by construction, its preimage line $\Re u = W/2$ is a hyperbolic geodesic in the strip. Translating back from u to w and then projecting by $z = e^w$ shows that C_* is a hyperbolic geodesic in A. Simplicity is clear because $|z| = \sqrt{rR}$ is an embedded circle.

To compute the length, evaluate the strip density on the center line. On S the hyperbolic density is

$$\rho_S(w) = \frac{\pi}{W}\csc\Big(\frac{\pi}{W}(\Re w - a)\Big),$$

so on $\Re w = a + W/2$ it becomes

$$\rho_S = \frac{\pi}{W}\csc\Big(\frac{\pi}{2}\Big) = \frac{\pi}{W}.$$

A single period of the closed loop corresponds to $w(t) = a + W/2 + it$ for $t \in [0, 2\pi]$. Then $|dw| = dt$, and the hyperbolic length is

$$\ell_A(C_*) = \int_0^{2\pi} \rho_S(w(t))\,dt = \int_0^{2\pi} \frac{\pi}{W}\,dt = \frac{2\pi^2}{W}.$$

Since $W = 2\pi m$, this equals $\ell_A(C_*) = \pi/m = \pi/\operatorname{mod}(A)$.

For minimality, observe that any simple closed curve in A homotopic to a boundary component lifts to a curve in the strip whose endpoints differ by $2\pi i$. In the strip, the density depends only on the real coordinate and is minimized when the cosecant factor is minimized, which happens at $\Re w = a + W/2$ where the argument is $\pi/2$. Any such lifted curve that deviates from this center line spends time at real parts where the density is larger, so its hyperbolic length cannot be smaller than the length of the center-line geodesic. This is the standard "shortest representative in a free homotopy class is a geodesic" phenomenon in hyperbolic geometry, specialized to the explicit strip model [34, 30]. □

Remark 5.17 (A clean modulus-controlled invariant). The identity

$$\ell_A(C_*) = \frac{\pi}{\operatorname{mod}(A)}$$

is one of the most concrete modulus-controls-geometry statements: the unique closed geodesic length is an explicit decreasing function of the modulus. Thin annulus ($\operatorname{mod} \to \infty$) means a very short core geodesic, while thick annulus ($\operatorname{mod} \to 0$) means a very long core geodesic.

Proposition 5.18 (Radial hyperbolic distance formula)**.** *Let $A = A(r, R)$, set $W = \log(R/r)$, and fix two radii $\rho_1, \rho_2 \in (r, R)$. Let $z_j = \rho_j e^{i\theta}$ for a fixed angle θ. Then the hyperbolic distance along the radial geodesic $\rho \mapsto \rho e^{i\theta}$ equals*

$$d_A(z_1, z_2) = \left| \log \left(\frac{\tan\left(\frac{\pi}{2W} \log(\rho_2/r)\right)}{\tan\left(\frac{\pi}{2W} \log(\rho_1/r)\right)} \right) \right|.$$

In particular, $d_A(z, \partial A) = \infty$ for every $z \in A$.

Proof. Take $w = \log z = \log \rho + i\theta$. A radial path $\rho \mapsto \rho e^{i\theta}$ corresponds to a horizontal segment $w(x) = x + i\theta$ in the strip, with $x \in (\log r, \log R)$. Hyperbolic length in A equals hyperbolic length in the strip under the covering. With $a = \log r$, the strip density is

$$\rho_S(x + i\theta) = \frac{\pi}{W} \csc\Big(\frac{\pi}{W}(x - a)\Big),$$

so the length from $x_1 = \log \rho_1$ to $x_2 = \log \rho_2$ is

$$d_A(z_1, z_2) = \left| \int_{x_1}^{x_2} \rho_S(x + i\theta)\, dx \right| = \left| \frac{\pi}{W} \int_{x_1}^{x_2} \csc\Big(\frac{\pi}{W}(x - a)\Big)\, dx \right|.$$

Set $u = \frac{\pi}{W}(x - a)$ so that $dx = \frac{W}{\pi}\, du$. When $x = x_j$ this gives $u_j = \frac{\pi}{W} \log(\rho_j/r)$, and the integral becomes

$$\int_{u_1}^{u_2} \csc(u)\, du.$$

An antiderivative is $\log|\tan(u/2)|$, which can be checked by differentiation, hence

$$\int_{u_1}^{u_2} \csc(u)\, du = \log \left| \frac{\tan(u_2/2)}{\tan(u_1/2)} \right|.$$

Substituting $u_j = \frac{\pi}{W} \log(\rho_j/r)$ yields the formula in the statement.

As $\rho \downarrow r$, we have $u \downarrow 0$ and $\tan(u/2) \sim u/2 \downarrow 0$, so the logarithm tends to $-\infty$ and the distance to the inner boundary diverges. As $\rho \uparrow R$, we have $u \uparrow \pi$ and $\tan(u/2) \to \infty$, so the distance to the outer boundary diverges. □

Remark 5.19 (Where modulus enters distances). In the distance formula the modulus enters only through $W = \log(R/r) = 2\pi \operatorname{mod}(A)$. Holding ρ_1 and ρ_2 fixed as fractions of the logarithmic coordinate, changing $\operatorname{mod}(A)$ rescales the argument of the tangent and therefore changes distances in an explicitly controlled way. This is the same mechanism that later produces modulus-dependent Schwarz–Pick derivative bounds and Lipschitz estimates.

6. Schwarz–Pick type control for maps defined on an annulus

6.1. **Schwarz–Pick lemma in metric form.** We now connect geometry (the hyperbolic metric) to analysis (derivative and distortion bounds) via Schwarz–Pick. This is the analytic core behind many "modulus controls holomorphic maps" statements. The disk version is classical, and the metric formulation on general hyperbolic domains is standard in complex analysis and hyperbolic geometry; see [12, 30, 33, 3].

Definition 6.1 (Hyperbolic metric density and infinitesimal norm)**.** Let Ω be a hyperbolic domain with hyperbolic density ρ_Ω. For $z \in \Omega$ and a tangent vector $v \in \mathbb{C}$ at z, define the hyperbolic norm

$$\|v\|_{\Omega,z} := \rho_\Omega(z)\,|v|.$$

Equivalently, the hyperbolic line element is $ds_\Omega = \rho_\Omega(z)|dz|$.

Definition 6.2 (Contraction in the hyperbolic metric)**.** Let $f : \Omega \to \Omega'$ be holomorphic between hyperbolic domains. We say f is infinitesimally nonexpanding if for every $z \in \Omega$ and every $v \in \mathbb{C}$,

$$\|f'(z) \cdot v\|_{\Omega',f(z)} \le \|v\|_{\Omega,z}.$$

In coordinates this is the pointwise inequality

$$\rho_{\Omega'}(f(z))\,|f'(z)| \le \rho_\Omega(z).$$

Theorem 6.3 (Schwarz–Pick in metric form)**.** *Let Ω, Ω' be hyperbolic domains with hyperbolic densities $\rho_\Omega, \rho_{\Omega'}$. If $f \in \mathrm{Hol}(\Omega, \Omega')$, then for all $z \in \Omega$,*

$$\rho_{\Omega'}(f(z))\,|f'(z)| \le \rho_\Omega(z).$$

Consequently, for all $z_1, z_2 \in \Omega$,

$$d_{\Omega'}(f(z_1), f(z_2)) \le d_\Omega(z_1, z_2).$$

Proof. Fix $z \in \Omega$. Choose universal covering maps $\pi : \mathbb{D} \to \Omega$ and $\pi' : \mathbb{D} \to \Omega'$ such that $\pi(0) = z$ and $\pi'(0) = f(z)$. Because $\mathbb{D}$ is simply connected, the map $f \circ \pi : \mathbb{D} \to \Omega'$ admits a holomorphic lift $\widetilde{f} : \mathbb{D} \to \mathbb{D}$ with $\pi' \circ \widetilde{f} = f \circ \pi$ and $\widetilde{f}(0) = 0$. The classical Schwarz lemma applied to $\widetilde{f}$ yields $|\widetilde{f}'(0)| \le 1$.

Differentiate $\pi' \circ \widetilde{f} = f \circ \pi$ at 0. The chain rule gives

$$\pi''(\widetilde{f}(0))\,\widetilde{f}'(0) = f'(\pi(0))\,\pi'(0),$$

so, using $\widetilde{f}(0) = 0$ and $\pi(0) = z$,

$$\pi''(0)\,\widetilde{f}'(0) = f'(z)\,\pi'(0).$$

Taking absolute values and using $|\widetilde{f}'(0)| \le 1$ gives

$$|f'(z)| \le \frac{|\pi''(0)|}{|\pi'(0)|}.$$

We now relate $|\pi'(0)|$ and $|\pi''(0)|$ to the hyperbolic densities. The hyperbolic metric on Ω is the pushforward of the disk hyperbolic metric under π, meaning that for every $\zeta \in \mathbb{D}$ one has $\rho_\Omega(\pi(\zeta))\,|\pi'(\zeta)| = \rho_\mathbb{D}(\zeta)$, and similarly $\rho_{\Omega'}(\pi'(\zeta))\,|\pi''(\zeta)| = \rho_\mathbb{D}(\zeta)$. Evaluating at $\zeta = 0$ yields

$$\rho_\Omega(z)\,|\pi'(0)| = \rho_\mathbb{D}(0), \qquad \rho_{\Omega'}(f(z))\,|\pi''(0)| = \rho_\mathbb{D}(0).$$

Eliminating $\rho_\mathbb{D}(0)$ gives

$$\frac{|\pi''(0)|}{|\pi'(0)|} = \frac{\rho_\Omega(z)}{\rho_{\Omega'}(f(z))},$$

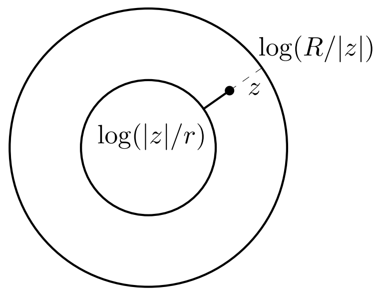


Figure 7. The logarithmic boundary distance $\Delta_A(z)$ is the *nearest* strip-distance in log coordinates. In Schwarz–Pick estimates one combines $\rho_{\mathbb{D}}(f(z))\,|f'(z)| \le \rho_A(z)$ with $\rho_A(z) \asymp \big(|z|\Delta_A(z)\big)^{-1}$ to convert intrinsic control into Euclidean derivative bounds.

and therefore

$$|f'(z)| \le \frac{\rho_\Omega(z)}{\rho_{\Omega'}(f(z))}, \qquad \rho_{\Omega'}(f(z))\,|f'(z)| \le \rho_\Omega(z).$$

This proves the infinitesimal inequality.

For the distance contraction, let γ be a rectifiable curve in Ω joining z_1 to z_2. Then $f \circ \gamma$ joins $f(z_1)$ to $f(z_2)$ and its hyperbolic length satisfies

$$\ell_{\Omega'}(f \circ \gamma) = \int_\gamma \rho_{\Omega'}(f(z))\,|f'(z)|\,|dz| \le \int_\gamma \rho_\Omega(z)\,|dz| = \ell_\Omega(\gamma).$$

Taking the infimum over all such γ yields $d_{\Omega'}(f(z_1), f(z_2)) \le d_\Omega(z_1, z_2)$. This is the standard metric form of Schwarz–Pick; see [12, 30, 33, 3]. $\square$

Remark 6.4 (Equality case). In the disk, equality in Schwarz–Pick is attained exactly by automorphisms of $\mathbb{D}$. In general hyperbolic domains, equality at one point with nonzero derivative forces f to be a local hyperbolic isometry, and when combined with properness it forces f to be a covering map onto its image. For annuli, this ties directly to the classification of proper maps later in Section 7 [30, 4].

6.2. **Quantitative derivative bounds on annuli.** We now specialize Theorem 6.3 to $X = A(r,R)$ and targets $Y = \mathbb{D}$ or another annulus. The point is that in Section 5 we obtained explicit control of ρ_A in terms of the modulus and a simple boundary-distance function. We emphasize that the universal inequality is independent of modulus; the modulus enters only when the domain density is made explicit.

Definition 6.5 (Disk hyperbolic density)**.** On $\mathbb{D}$ we use the standard hyperbolic density

$$\rho_{\mathbb{D}}(\zeta) = \frac{2}{1 - |\zeta|^2}.$$

Theorem 6.6 (Pointwise derivative control: annulus $\to$ disk)**.** *Let $A = A(r,R)$ and set $W = \log(R/r) = 2\pi \mathrm{mod}(A)$. Let $f \in \mathrm{Hol}(A, \mathbb{D})$. Then for every $z \in A$,*

$$|f'(z)| \le \frac{\rho_A(z)}{\rho_{\mathbb{D}}(f(z))} = \frac{1 - |f(z)|^2}{2}\,\rho_A(z).$$

Using the explicit density from Theorem 5.8, this becomes

$$|f'(z)| \le \frac{1-|f(z)|^2}{2} \cdot \frac{\pi}{W} \cdot \frac{1}{|z|} \cdot \csc\Big(\frac{\pi}{W}\log\Big(\frac{|z|}{r}\Big)\Big).$$

Using the cylindrical comparison Proposition 5.13, one also has the clean estimate

$$|f'(z)| \le \frac{1-|f(z)|^2}{2} \cdot \frac{\pi}{2} \cdot \frac{1}{|z|\Delta_A(z)},$$

where $\Delta_A(z) = \min\{\log(|z|/r), \log(R/|z|)\}$.

Proof. Apply Theorem 6.3 with $\Omega = A$ and $\Omega' = \mathbb{D}$ to obtain $\rho_{\mathbb{D}}(f(z))\,|f'(z)| \le \rho_A(z)$. Divide by $\rho_{\mathbb{D}}(f(z))$ and then substitute $\rho_{\mathbb{D}}(\zeta) = 2/(1-|\zeta|^2)$ to get $|f'(z)| \le \frac{1-|f(z)|^2}{2}\rho_A(z)$. Inserting the explicit formula for $\rho_A(z)$ from Theorem 5.8 yields the displayed trigonometric expression. Finally, Proposition 5.13 gives $\rho_A(z) \le \frac{\pi}{2} \cdot \frac{1}{|z|\Delta_A(z)}$, and substituting this upper bound into the previous inequality yields the boundary-distance form. □

Corollary 6.7 (Uniform derivative bound away from the boundary)**.** *Let* $A = A(r,R)$, $W = \log(R/r)$, *and let* $0 < \varepsilon < W/2$. *Define the* ε*-core subannulus*

$$A_\varepsilon := \Big\{ z \in A : \ \varepsilon \le \log\Big(\frac{|z|}{r}\Big) \le W - \varepsilon \Big\}.$$

Then for every $f \in \mathrm{Hol}(A, \mathbb{D})$ *and every* $z \in A_\varepsilon$,

$$|f'(z)| \le \frac{1}{2} \cdot \frac{\pi}{W} \cdot \frac{1}{|z|} \cdot \csc\Big(\frac{\pi\varepsilon}{W}\Big).$$

The dependence on ε *enters only through the normalized depth* ε/W *and thus is an explicit function of* $\mathrm{mod}(A)$ *once* ε *is specified as a fraction of the strip width.*

Proof. Since $f(z) \in \mathbb{D}$, we have $1 - |f(z)|^2 \le 1$, so Theorem 6.6 implies $|f'(z)| \le \frac{1}{2}\rho_A(z)$. Write $t = \log(|z|/r)$. If $z \in A_\varepsilon$, then $t \in [\varepsilon, W-\varepsilon]$, so $\frac{\pi t}{W} \in [\frac{\pi\varepsilon}{W}, \pi - \frac{\pi\varepsilon}{W}]$ and therefore $\sin(\frac{\pi t}{W}) \ge \sin(\frac{\pi\varepsilon}{W})$. Equivalently, $\csc(\frac{\pi t}{W}) \le \csc(\frac{\pi\varepsilon}{W})$. Substitute into the explicit formula $\rho_A(z) = \frac{\pi}{W|z|}\csc(\frac{\pi t}{W})$ to obtain $\rho_A(z) \le \frac{\pi}{W|z|}\csc(\frac{\pi\varepsilon}{W})$, then multiply by $1/2$. □

Corollary 6.8 (Hyperbolic Lipschitz control)**.** *Let* A *be a nondegenerate annulus and* $f \in \mathrm{Hol}(A, \mathbb{D})$. *Then* f *is* 1*-Lipschitz for hyperbolic distances:*

$$d_{\mathbb{D}}(f(z_1), f(z_2)) \le d_A(z_1, z_2) \qquad \text{for all } z_1, z_2 \in A.$$

Proof. This is the distance contraction statement in Theorem 6.3. □

Remark 6.9 (Where the modulus enters). Schwarz–Pick itself is universal. The modulus dependence enters through ρ_A. For $A(r,R)$,

$$\rho_A(z) = \frac{\pi}{W|z|}\csc\Big(\frac{\pi}{W}\log\Big(\frac{|z|}{r}\Big)\Big), \qquad W = 2\pi\mathrm{mod}(A),$$

and Proposition 5.13 converts this into the clean boundary-distance scale $\rho_A(z) \asymp \frac{1}{|z|\Delta_A(z)}$ with absolute constants. Thus all constants in derivative and distortion bounds become explicit functions of $\mathrm{mod}(A)$ once the geometric location of z inside the annulus is fixed in strip coordinates.

6.3. **Distortion and growth for holomorphic functions on thin vs thick annuli.** We now interpret the bounds in the two asymptotic regimes $\mathrm{mod}(A) \to \infty$ (thin annuli) and $\mathrm{mod}(A) \to 0$ (thick annuli). The analytic input is the exponential mode suppression from Section 4 and the explicit hyperbolic density from Section 5. The combination is precisely what turns qualitative Schwarz–Pick contraction into quantitative statements with rates.

Remark 6.10 (Computational experiments). We implemented three numerical experiments in a companion notebook.

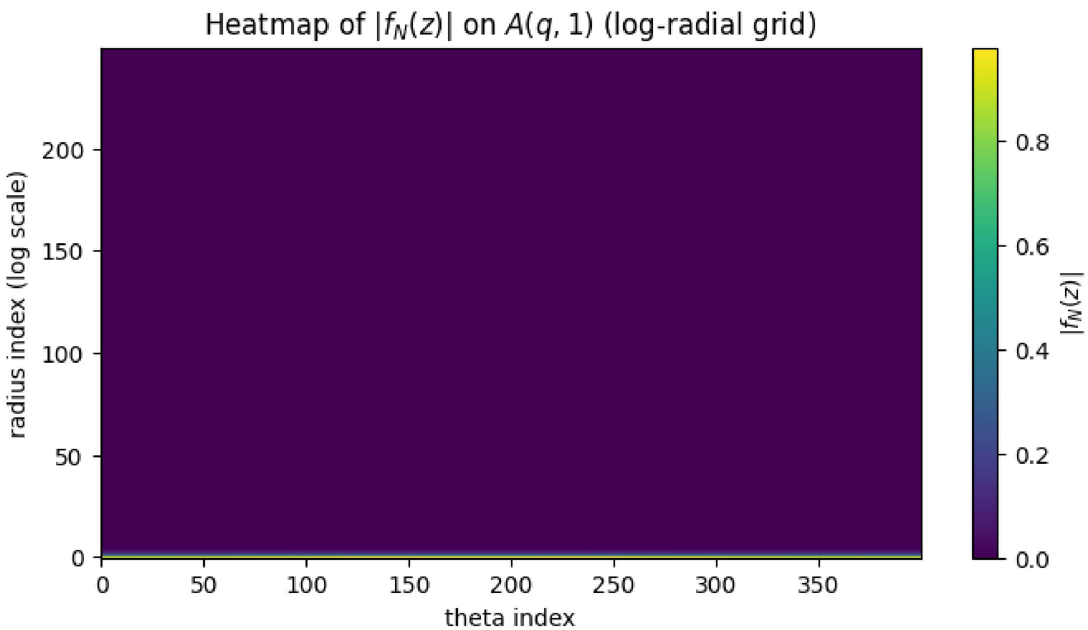


Figure 8. Heatmap of $|f_N(z)|$ on $A(q, 1)$ using a log-radial grid. The function $f_N(z) = \sum_{|n|\le N} a_n z^n$ is generated with random complex coefficients and then rescaled by boundary sampling so that $\|f_N\|_{L^\infty(A(q,1))} \approx 1$.

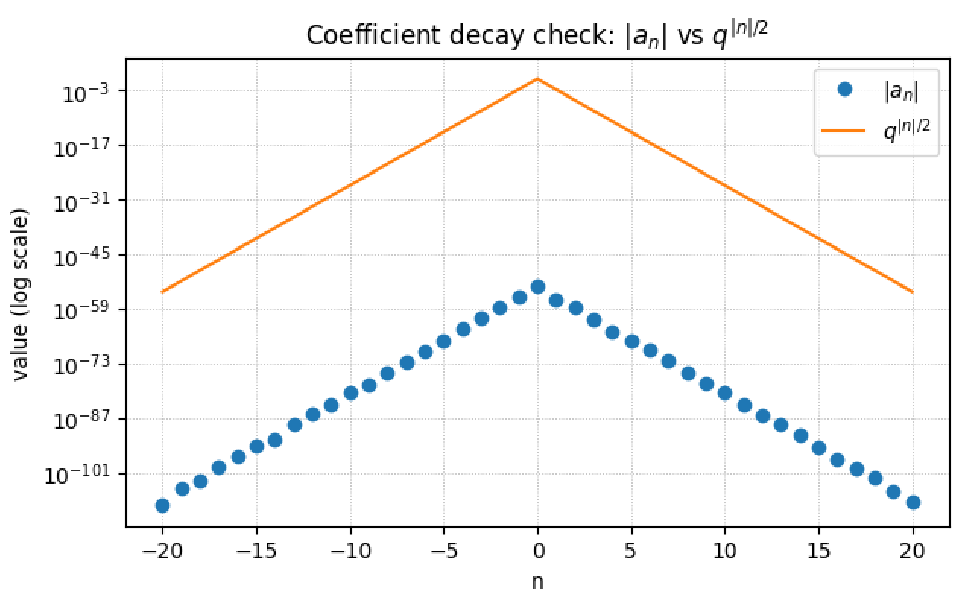


Figure 9. Coefficient decay check on $A(q,1)$: the sampled coefficients $|a_n|$ (blue) compared to the envelope $q^{|n|/2}$ from Theorem 4.11. Typical samples lie below the worst-case bound because the sampling uses an additional decay envelope and then rescales to satisfy the boundary constraint.

Reproducibility. The exact source code used to generate Experiment 1 (Figures 8–9) is recorded in Appendix H.1.

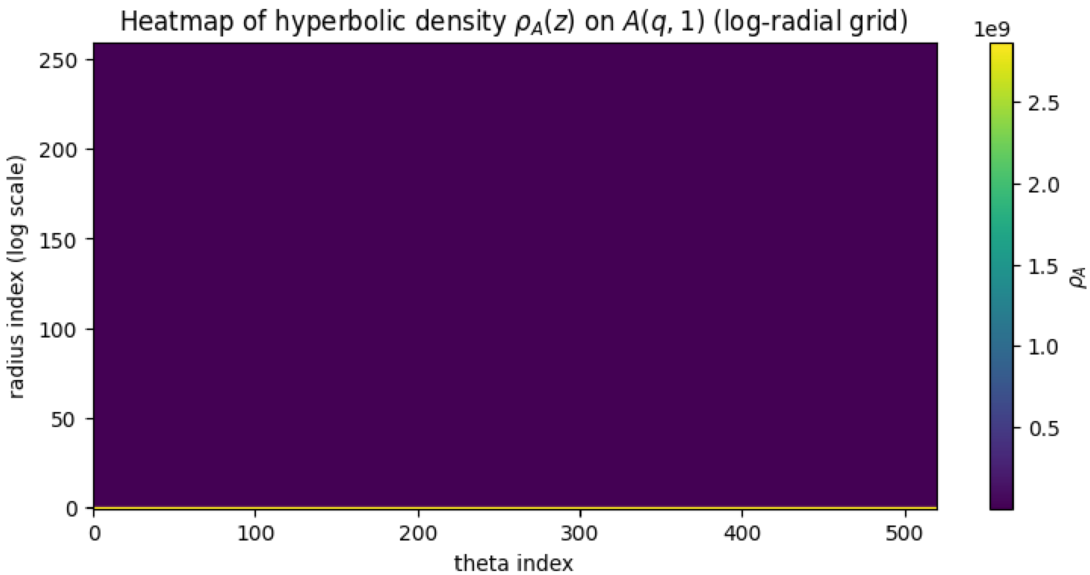


Figure 10. Heatmap of the hyperbolic density $\rho_A(z)$ on $A(q,1)$ evaluated on a log-radial grid. The blow-up near the boundary components matches Theorem 5.8 and the completeness discussion in Remark 5.9.

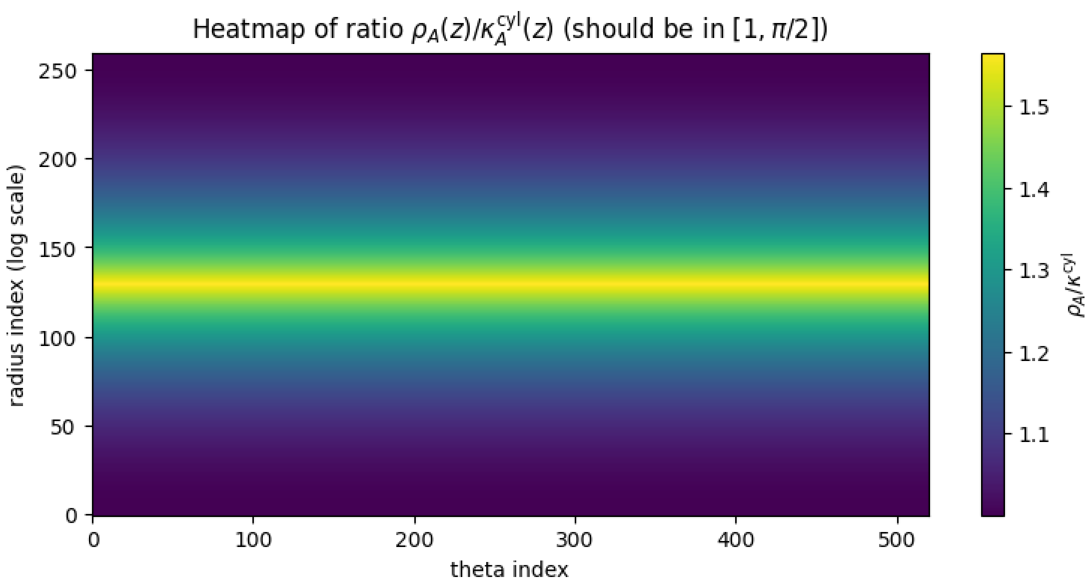


Figure 11. Heatmap of the ratio $\rho_A(z)/\kappa_A^{\mathrm{cyl}}(z)$ on $A(q,1)$, where $\kappa_A^{\mathrm{cyl}}(z) = 1/(|z|\Delta_A(z))$. Proposition 5.13 predicts $1 \le \rho_A/\kappa_A^{\mathrm{cyl}} \le \pi/2$.

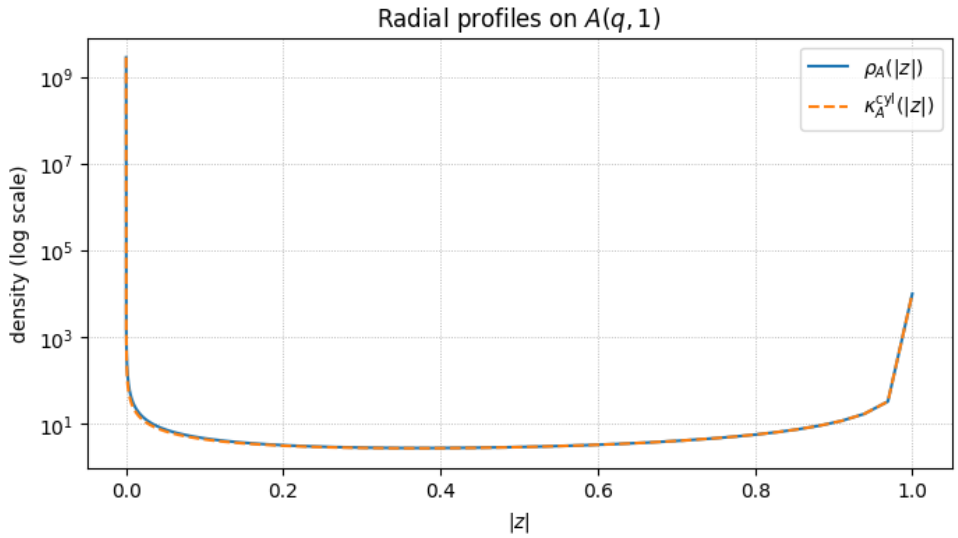


Figure 12. Radial profiles (log scale) comparing $\rho_A(|z|)$ to $\kappa_A^{\mathrm{cyl}}(|z|)$, illustrating the two-sided comparability of Proposition 5.13.

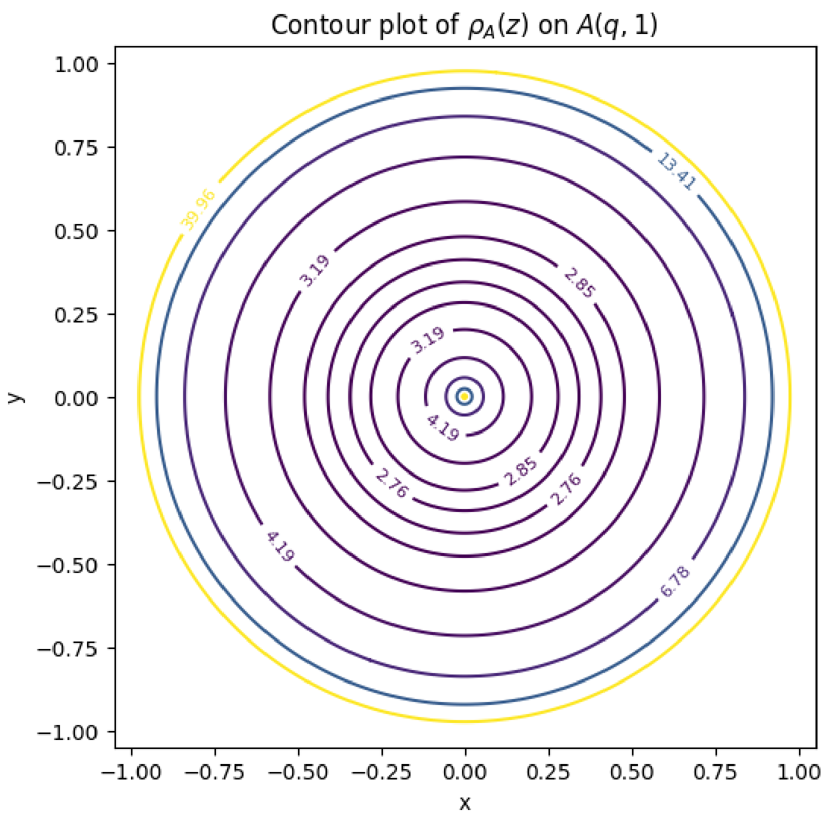


Figure 13. Contour plot of $\rho_A(z)$ in the Euclidean plane (masked outside $A(q, 1)$). In the round model ρ_A is radial, so level sets are circles; this plot is a geometric sanity check.

Reproducibility. The source code used to generate Experiment 2 (Figures 10–13) is recorded in Appendix H.2.

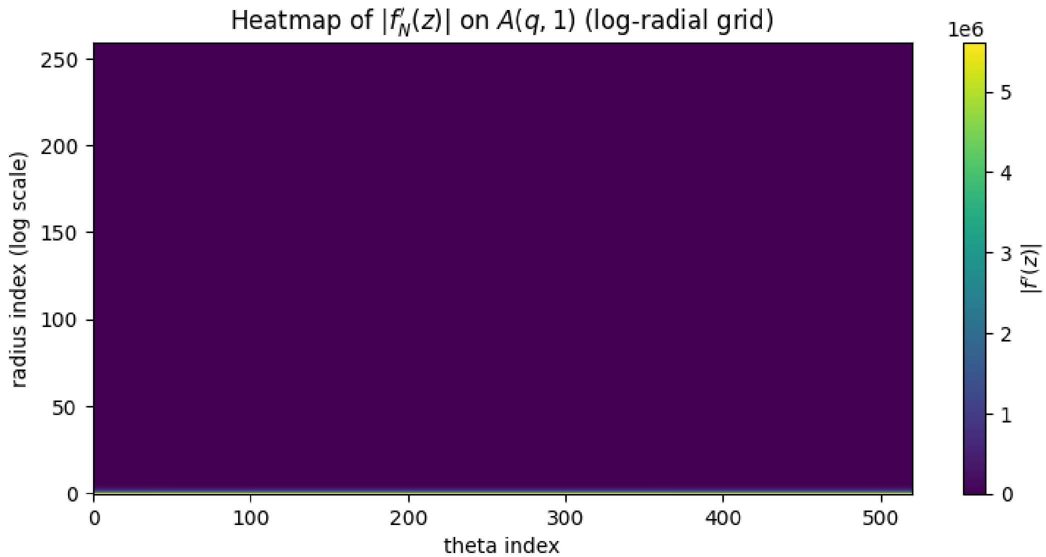


Figure 14. Radial slice comparison at a fixed angle: $|f_N'(z)|$ (blue) versus the pointwise bound $(1/2)\rho_A(z)$ and the uniform interior bound on A_ε from Corollary 6.7.

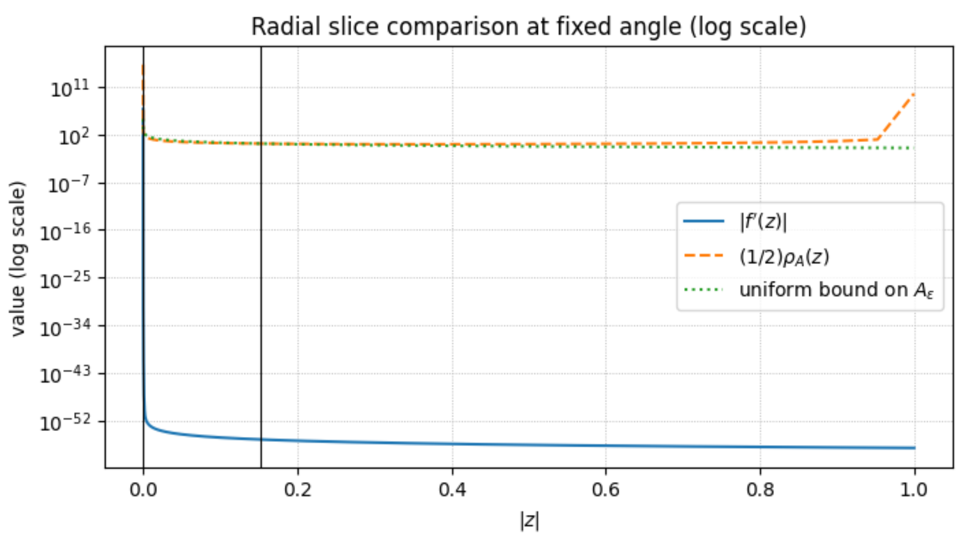


Figure 15. Distribution over trials of $\sup_{z\in A_\varepsilon} |f_N'(z)|$ for boundary-normalized random Laurent polynomials.

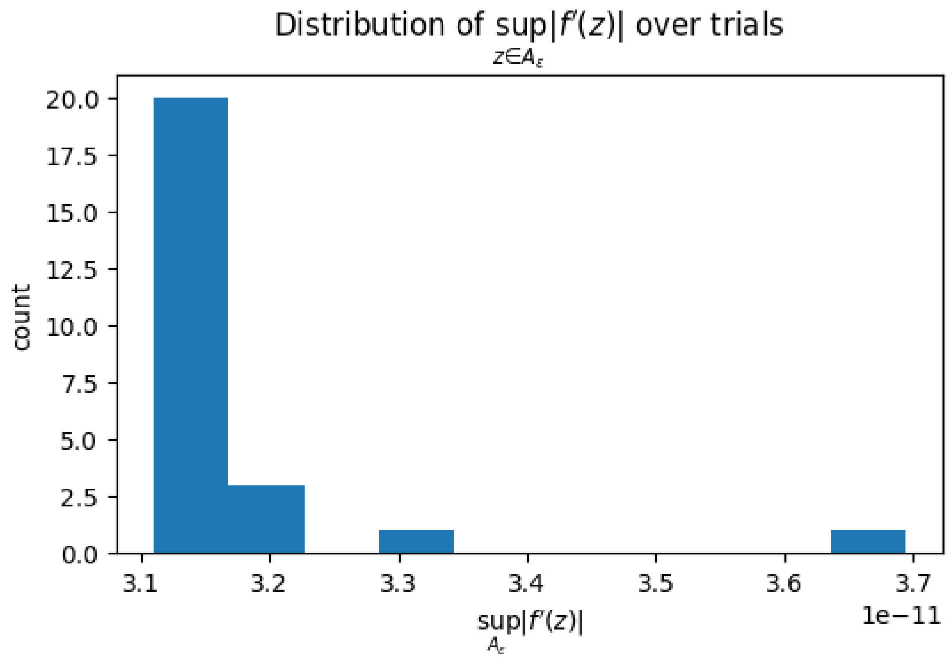


Figure 16. Distribution of $\max_{z\in A_\varepsilon} |f'_N(z)|/((1/2)\rho_A(z))$ computed on the sampling grid, as a diagnostic relative to the curvature-normalized scale ρ_A.

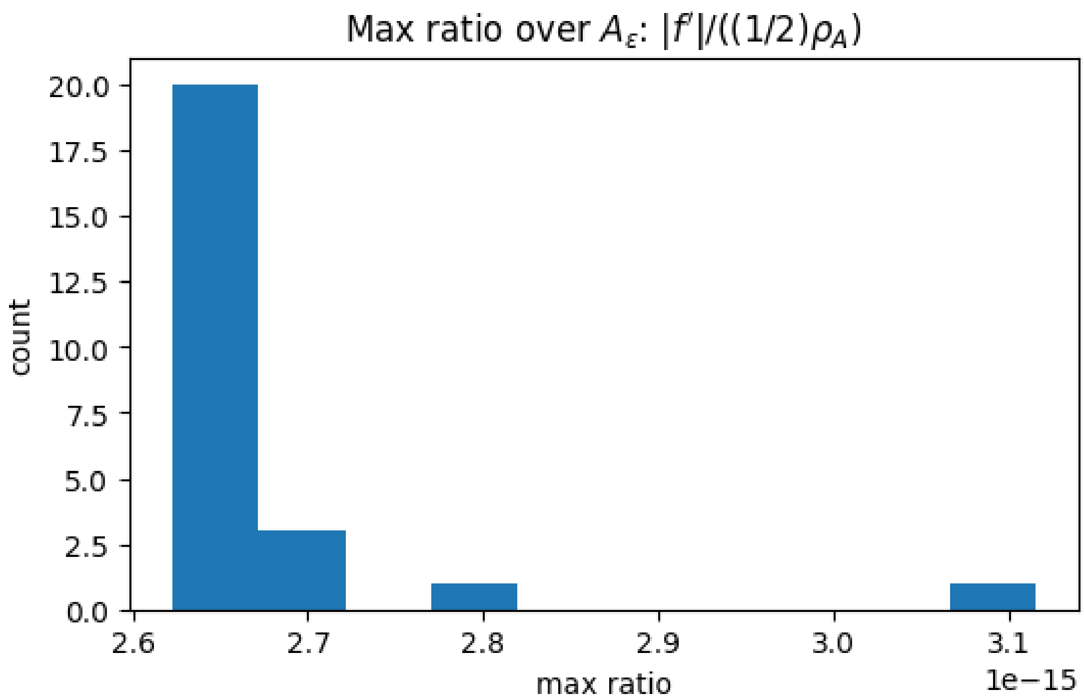


Figure 17. Heatmap of $|f'_N(z)|$ on $A(q,1)$ (log-radial grid). Peaks near the boundary reflect boundary blow-up scale; the ε-core remains comparatively flat.

Reproducibility. The source code used to generate Experiment 3 (Figures 14–17) is recorded in Appendix H.3.

Proposition 6.11 (Thin annuli enforce angular rigidity for bounded maps)**.** *Let $A(q,1)$ be a normalized annulus with $q = e^{-2\pi m}$, so $\operatorname{mod}(A) = m$. Let $f \in \operatorname{Hol}(A(q,1), \mathbb{D})$ and write its Laurent series*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n.$$

Then for each $n \neq 0$,

$$|a_n| \le q^{|n|/2} = \exp(-\pi|n|\, m).$$

In particular, as $m \to \infty$ the nonconstant modes vanish exponentially in m, and on the mid-circle $|z| = \sqrt{q}$ the function becomes exponentially close to the constant mode a_0 as quantified in Corollary 4.13.

Proof. Since $|f| \le 1$ on $A(q,1)$, Theorem 4.11 with $M = 1$ yields $|a_n| \le q^{|n|/2}$ for all $n \in \mathbb{Z}$. This gives the claimed inequality and the exponential interpretation follows from $q = e^{-2\pi m}$. □

Remark 6.12 (Interpretation in strip coordinates)*.* On the strip $S = \{0 < \Re w < 2\pi m\}$, the lift $F(w) = f(e^w)$ has Fourier series

$$F(w) = \sum_{n\in\mathbb{Z}} a_n e^{nw},$$

and the factor $\exp(-\pi|n|m)$ is exactly exponential decay in the strip width. Thus for large modulus, F is dominated by the $n = 0$ mode on the midline $\Re w = \pi m$, making F nearly independent of $\Im w$ (the angular variable). This is the analytic mechanism behind thin-annulus rigidity statements and is closely aligned with the cylinder picture from Section 4.

Proposition 6.13 (Boundary blow-up scale for derivatives)**.** *Let $A = A(r,R)$ with width $W = \log(R/r)$. Let $f \in \operatorname{Hol}(A, \mathbb{D})$. Then for each $z \in A$,*

$$|f'(z)| \le \frac{\pi}{4} \cdot \frac{1}{|z|\Delta_A(z)}, \qquad \Delta_A(z) = \min\Big\{ \log\Big(\frac{|z|}{r}\Big), \log\Big(\frac{R}{|z|}\Big)\Big\}.$$

As z approaches either boundary component, $\Delta_A(z) \to 0$ and the bound diverges at the same scale as the Euclidean boundary distance in the sense that locally

$$\frac{1}{|z|\Delta_A(z)} \asymp \frac{1}{\operatorname{dist}(z, \partial A)}.$$

Proof. Theorem 6.6 gives $|f'(z)| \le \frac{1}{2}\rho_A(z)$ because $1 - |f(z)|^2 \le 1$. Proposition 5.13 gives $\rho_A(z) \le \frac{\pi}{2} \cdot \frac{1}{|z|\Delta_A(z)}$. Multiplying yields

$$|f'(z)| \le \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{1}{|z|\Delta_A(z)} = \frac{\pi}{4} \cdot \frac{1}{|z|\Delta_A(z)}.$$

Near $|z| = r$, write $|z| = r + \varepsilon$ with $\varepsilon \downarrow 0$. Then $\log(|z|/r) = \log(1 + \varepsilon/r) \sim \varepsilon/r$ and $|z| \sim r$, so $|z|\Delta_A(z) \sim r \cdot (\varepsilon/r) = \varepsilon = \operatorname{dist}(z, \partial A)$. The outer boundary case is analogous. □

Remark 6.14 (Thick annuli and the $1/\operatorname{mod}(A)$ scale). When $\operatorname{mod}(A) \to 0$, the width $W = \log(R/r) \to 0$ and the explicit density

$$\rho_A(z) = \frac{\pi}{W|z|} \csc\Big(\frac{\pi}{W} \log\Big(\frac{|z|}{r}\Big)\Big)$$

contains the large prefactor $\pi/W \sim 1/\operatorname{mod}(A)$. Even at the geometric center $|z| = \sqrt{rR}$, the cosecant term equals 1, so ρ_A itself is of size $\pi/(W|z|)$. Thus Euclidean derivative bounds extracted from Schwarz–Pick typically grow like $1/W \sim 1/\operatorname{mod}(A)$ on thick annuli. This is the same scale on which the core geodesic length $\ell_A(C_*) = \pi/\operatorname{mod}(A)$ grows in Section 5.

6.4. **Examples and sharpness.** We record canonical examples that attain or nearly attain Schwarz–Pick type bounds, and we connect them to later rigidity statements about proper maps between annuli.

Example 6.15 (Rotations are hyperbolic isometries)**.** Any automorphism of a hyperbolic domain is a hyperbolic isometry and therefore attains equality in Schwarz–Pick. On $A(r,R)$, the rotations $f(z) = e^{i\theta}z$ satisfy $|f'(z)| = 1$ and $\rho_A(f(z)) = \rho_A(z)$ because ρ_A is radial, so

$$\rho_A(f(z))\,|f'(z)| = \rho_A(z).$$

Example 6.16 (Power maps between annuli)**.** Fix $n \in$ and consider $f(z) = z^n$. Then $f : A(r,R) \to A(r^n, R^n)$ and the moduli satisfy

$$\mathrm{mod}(A(r^n,R^n)) = \frac{1}{2\pi}\log\left(\frac{R^n}{r^n}\right) = n\cdot\frac{1}{2\pi}\log\left(\frac{R}{r}\right) = n\,\mathrm{mod}(A(r,R)).$$

In strip coordinates, $F(w) = f(e^w) = e^{nw}$ is a single Fourier mode, and the induced map on the imaginary period is multiplication by n. This is the prototype for the general modulus scaling law for proper maps proved later in Section 7; compare [4, 2].

Example 6.17 (Extremals via universal covering lifts)**.** Let $\pi : \mathbb{D} \to A(r,R)$ be a universal covering map. Composing π with disk automorphisms produces many sharp situations because the proof of Schwarz–Pick factors through a disk self-map $\widetilde{f}$ on the universal cover. Conversely, if one wishes to show an inequality on A is sharp, one often constructs f by prescribing $\widetilde{f}$ on $\mathbb{D}$ and then descending through the covering relationship $\pi' \circ \widetilde{f} = f \circ \pi$ [12, 30].

Remark 6.18 (Blaschke products in strip coordinates). In strip coordinates, periodic disk-valued maps can be built from a disk variable

$$\zeta = \exp\Big(\frac{2\pi i}{W}w\Big),$$

which maps the strip to $\mathbb{D}$ and respects the $2\pi i$-periodicity. Finite Blaschke products in ζ can then be composed with $\zeta(w)$ and pushed down to the annulus. This produces explicit families $f : A(r,R) \to \mathbb{D}$ whose oscillation and derivative behavior depend on $W = 2\pi\mathrm{mod}(A)$. We keep formulas implicit here and return to them if we need a fully explicit extremizer for a particular bound [3, 7].

7. Proper holomorphic maps between annuli and modulus scaling laws

7.1. **Topological degree and induced map on π_1.** We now explain why the modulus is not merely a geometric parameter but a mapping constraint. The key bridge is the interaction between holomorphy, properness, and the fundamental group $\pi_1(A) \cong \mathbb{Z}$. The general background facts we use about coverings, proper holomorphic maps, and fundamental groups can be found in standard references such as [4, 2, 3].

Definition 7.1 (Proper map)**.** Let X, Y be Hausdorff topological spaces. A continuous map $f : X \to Y$ is proper if the preimage of every compact set is compact, meaning that for each compact $K \subset Y$ one has $f^{-1}(K) \subset X$ compact.

Remark 7.2 (Properness for maps between bounded planar domains). If $X \subset \mathbb{C}$ and $Y \subset \mathbb{C}$ are planar domains and $f : X \to Y$ is continuous, then f is proper if and only if every sequence $z_k \in X$ converging to ∂X satisfies $f(z_k)$ converging to ∂Y. This is the precise version of the boundary-to-boundary intuition; see [4, 3].

Definition 7.3 (Degree for maps between annuli)**.** Let A, A' be annuli. Fix a positively oriented simple closed curve γ in A representing the generator of $\pi_1(A) \cong \mathbb{Z}$ (for example, a core circle when A is round). For a continuous map $f : A \to A'$, define its degree as the integer $n \in \mathbb{Z}$ such that the induced homomorphism

$$f_* : \pi_1(A) \to \pi_1(A')$$

is multiplication by n under the identifications $\pi_1(A) \cong \mathbb{Z} \cong \pi_1(A')$. We write $\deg(f) := n$.

Lemma 7.4 (Degree via winding around 0 in the round target)**.** *Let A be an annulus and let $A' = A(r', R')$ be a round annulus, so $0 \notin A'$. Let $f : A \to A'$ be continuous and let γ be a generator loop in A. Then $\deg(f)$ equals the winding number of the closed curve $f \circ \gamma$ around 0, namely*

$$\deg(f) = \frac{1}{2\pi} \Delta_\gamma \arg(f(\gamma(t))).$$

If f is holomorphic and nonvanishing on A, then equivalently

$$\deg(f) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

Proof. Because $A' \subset \mathbb{C} \setminus \{0\}$ is doubly connected, its fundamental group is generated by a loop winding once around 0, and the identification $\pi_1(A') \cong \mathbb{Z}$ is precisely the winding number around the origin. By definition, $\deg(f)$ is the integer describing how f sends the generator of $\pi_1(A)$ to $\pi_1(A')$, which is the same integer as the winding number of the loop $f \circ \gamma$ around 0.

If f is holomorphic and nonvanishing on a neighborhood of γ, one may choose a continuous branch of $\log f$ along γ. Differentiating gives $(\log f)' = f'/f$ along the curve, and integrating around the loop yields the total change in $\log f$, whose imaginary part is the total change in $\arg f$. Dividing by 2π gives the winding number, producing the integral formula. This is the standard variation-of-argument identity, a local form of the argument principle [3, 12]. □

Lemma 7.5 (Proper holomorphic maps between annuli are finite-sheeted coverings)**.** *Let $f : A \to A'$ be a nonconstant proper holomorphic map between annuli. Then f is a finite branched covering map onto A' in the sense that each point $w \in A'$ has finitely many preimages in A, and outside the discrete critical set $\{f' = 0\}$ the map is a genuine covering map. In particular, $f_* : \pi_1(A) \to \pi_1(A')$ is injective, hence $\deg(f) \neq 0$.*

Proof. A nonconstant holomorphic map is open, so $f(A)$ is an open subset of A'. Properness implies $f(A)$ is also closed in A' because if $w_j \in f(A)$ converges to $w \in A'$, pick $z_j \in A$ with $f(z_j) = w_j$. The set $\{w_j\} \cup \{w\}$ is compact in A', so its preimage under a proper map is compact in A and thus contains a convergent subsequence $z_{j_k} \to z_\infty \in A$. Continuity gives $f(z_\infty) = w$, so $w \in f(A)$ and $f(A)$ is closed. Since A' is connected, $f(A) = A'$.

Fix $w \in A'$. Properness implies $f^{-1}(\{w\})$ is compact. If it were infinite, it would have a limit point in A, which is impossible because zeros of $f(z) - w$ are isolated for a nonconstant holomorphic function. Hence each fiber is finite. Away from critical points, the holomorphic inverse function theorem makes f a local biholomorphism, hence a covering map over $A' \setminus f(\{f' = 0\})$. In particular, restricting to a loop avoiding the critical values shows f induces an injective map on π_1, which forces $\deg(f) \neq 0$ because $\mathbb{Z} \to \mathbb{Z}$ injective means multiplication by a nonzero integer. These are standard facts about proper holomorphic maps between planar domains [4, 3]. □

Proposition 7.6 (Proper holomorphic maps have nonzero degree and a natural sign convention)**.** *Let $f : A \to A'$ be a proper holomorphic map between annuli. Then $\deg(f) \neq 0$. If we compose the target with inversion $I(z) = 1/z$ when convenient, then we may assume $\deg(f) = n \in$.*

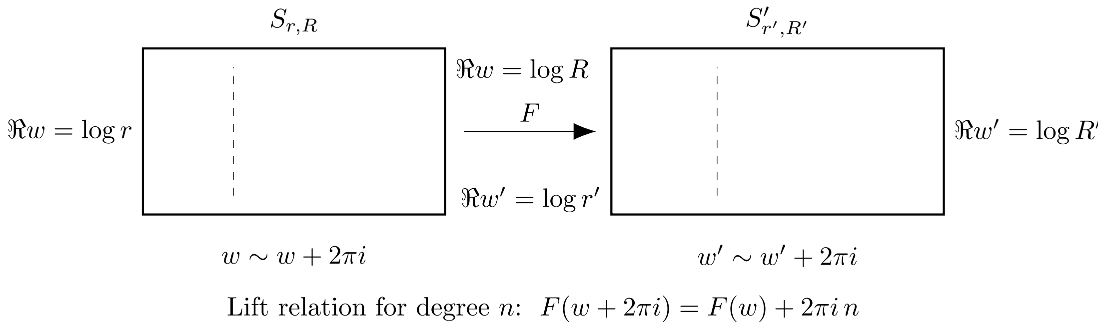


Figure 18. A proper map $f : A(r, R) \to A(r', R')$ lifts to a holomorphic map $F : S_{r,R} \to S'_{r',R'}$ between strips whose imaginary-period shift records the degree. This is the mechanism behind the monomial form and the modulus scaling law.

Proof. Lemma 7.5 gives $\deg(f) \neq 0$. The inversion map $I(z) = 1/z$ on any annulus reverses the preferred generator orientation on π_1 and therefore sends degree n to degree $-n$. Hence by composing with inversion if necessary, one may assume the degree is a positive integer. □

7.2. **Classification theorem for proper maps of round annuli.** We now prove the fundamental rigidity statement: proper holomorphic maps between round annuli are monomials up to a constant, and the modulus scaling law is forced. This theorem is classical and can be proved in several ways; the approach below is designed to make the modulus dependence completely explicit and to dovetail with our strip-covering formalism [2, 4, 3].

Theorem 7.7 (Classification of proper holomorphic maps between round annuli)**.** *Let $A = A(r, R)$ and $A' = A(r', R')$ be round annuli. Let $f : A \to A'$ be a proper holomorphic map with* $\deg(f) = n \in N$. *Then there exists a constant $c \in \mathbb{C} \setminus \{0\}$ such that*

$$f(z) = c\, z^n \qquad \textit{for all } z \in A.$$

Moreover,

$$\frac{R'}{r'} = \left(\frac{R}{r}\right)^n, \qquad \textit{equivalently} \qquad \operatorname{mod}(A') = n \operatorname{mod}(A).$$

If instead $\deg(f) = -n$*, then*

$$f(z) = c\, z^{-n}, \qquad \operatorname{mod}(A') = n \operatorname{mod}(A),$$

and composing with inversion reduces to the positive-degree case.

Proof. Let $S = \{w : \log r < \Re w < \log R\}$ and $S' = \{w' : \log r' < \Re w' < \log R'\}$ be the logarithmic strips. The exponential maps $\exp : S \to A$ and $\exp : S' \to A'$ are universal coverings with deck groups generated by $w \mapsto w + 2\pi i$ and $w' \mapsto w' + 2\pi i$. Since S is simply connected, the map $f \circ \exp : S \to A'$ admits a holomorphic lift $F : S \to S'$ with $\exp(F(w)) = f(e^w)$.

For each $w \in S$ one has

$$\exp(F(w + 2\pi i)) = f(e^{w+2\pi i}) = f(e^w) = \exp(F(w)),$$

so $F(w + 2\pi i) - F(w) \in 2\pi i\, \mathbb{Z}$. The function

$$k(w) := \frac{F(w + 2\pi i) - F(w)}{2\pi i}$$

is holomorphic on the connected domain S and takes values in the discrete set $\mathbb{Z}$, hence it is constant. Denote this constant by $k \in \mathbb{Z}$, so

$$F(w + 2\pi i) = F(w) + 2\pi i\, k \qquad (w \in S).$$

The integer k is exactly the induced map on π_1, hence $k = \deg(f) = n$ by Definition 7.3 and the strip interpretation of winding, so

$$F(w + 2\pi i) = F(w) + 2\pi i\, n.$$

Consider the holomorphic correction

$$G(w) := F(w) - nw.$$

The translation identity shows G is $2\pi i$-periodic:

$$G(w + 2\pi i) = F(w + 2\pi i) - n(w + 2\pi i) = F(w) + 2\pi i n - nw - 2\pi i n = G(w).$$

We claim G is constant.

Because $F(S) \subset S'$, the real part of F is bounded: $\log r' < \Re F(w) < \log R'$ for all $w \in S$. On the other hand, $\Re(nw) = n\Re w$ ranges over the finite interval $(n \log r, n \log R)$ as w varies in S. Hence $\Re G = \Re F - n\Re w$ is bounded above and below on S.

Now use periodicity to descend G to the annulus. Define

$$H(z) := G(\log z) \qquad \text{for } z \in A(r, R),$$

where $\log z$ denotes any logarithm $w \in S$ with $e^w = z$. This is well-defined because G is $2\pi i$-periodic. The function H is holomorphic on $A(r, R)$, and its real part is bounded because $\Re G$ is bounded on S. Therefore the holomorphic function $\exp(H)$ is bounded and bounded away from 0 on $A(r, R)$:

$$0 < \inf_{z \in A} |\exp(H(z))| \le \sup_{z \in A} |\exp(H(z))| < \infty.$$

A holomorphic function on an annulus whose modulus is bounded above and below cannot have zeros or poles and cannot develop essential boundary behavior; in particular, the maximum principle applied to $\exp(H)$ and to $1/\exp(H)$ forces $\exp(H)$ to be constant on each boundary component and hence constant throughout the annulus. One clean way to formalize this is to observe that $\log|\exp(H)| = \Re H$ is harmonic and bounded, so $\Re H$ has nontangential limits a.e. on both boundary circles, and by the maximum principle it must be constant if it attains both its supremum and infimum on the boundary; the bounded-above and bounded-below condition forces this situation for both $\exp(H)$ and $1/\exp(H)$. Equivalently, one may invoke the standard classification of nowhere-vanishing holomorphic functions on a round annulus: any such function is cz^m for some $m \in \mathbb{Z}$, and boundedness above and below forces $m = 0$. In either formulation, H is constant, hence G is constant on S. Write $G(w) \equiv \beta \in \mathbb{C}$.

Thus $F(w) = nw + \beta$, and descending through the exponential gives, for $z = e^w$,

$$f(z) = \exp(F(w)) = \exp(nw + \beta) = e^\beta (e^w)^n = c\, z^n, \qquad c := e^\beta \neq 0.$$

Finally, properness forces the modulus scaling by comparing boundary circles. Since $|f(z)| = |c|\,|z|^n$, the image of the circle $|z| = \rho$ is the circle $|w| = |c|\,\rho^n$. Hence

$$f(A(r, R)) = A(|c| r^n,\ |c| R^n).$$

Because f maps onto $A(r', R')$, we must have $r' = |c| r^n$ and $R' = |c| R^n$, and therefore

$$\frac{R'}{r'} = \left(\frac{R}{r}\right)^n.$$

Taking logarithms and dividing by 2π yields $\operatorname{mod}(A') = n \operatorname{mod}(A)$.

If $\deg(f) = -n$, the same argument runs with $F(w + 2\pi i) = F(w) - 2\pi i n$ and the periodic correction $G(w) = F(w) + nw$, producing $F(w) = -nw + \beta$ and hence $f(z) = cz^{-n}$, with the same modulus ratio. ☐

7.3. **Consequences and rigidity.** We now package the theorem into quick consequences that we will use later. The first is the sharp existence criterion: the modulus is the only obstruction, and it is arithmetic.

Corollary 7.8 (Existence constraint for proper maps)**.** *Let A, A' be nondegenerate annuli and let $n \in$. There exists a proper holomorphic map $f : A \to A'$ with* $\deg(f) = n$ *if and only if*

$$\operatorname{mod}(A') = n \operatorname{mod}(A).$$

Proof. Choose biholomorphisms $\phi : A \to A(r, R)$ and $\psi : A' \to A(r', R')$ by the conformal classification of annuli (Theorem 2.12). Then f is proper holomorphic of degree n if and only if $\psi \circ f \circ \phi^{-1}$ is a proper holomorphic map of degree n between the round models. Theorem 7.7 then forces $\operatorname{mod}(A(r', R')) = n \operatorname{mod}(A(r, R))$, which is exactly $\operatorname{mod}(A') = n\operatorname{mod}(A)$ by Definition 2.13. Conversely, if $\operatorname{mod}(A') = n\operatorname{mod}(A)$, pick round models with $R'/r' = (R/r)^n$ and define $f(z) = cz^n$ with $|c| = r'/r^n$. This gives a proper holomorphic map between the models, and conjugating by ϕ, ψ gives the desired $f : A \to A'$. ☐

Corollary 7.9 (Rigidity up to rotations and inversion)**.** *Let $f : A(r, R) \to A(r', R')$ be a proper holomorphic map with* $\deg(f) = n \in \mathbb{Z} \setminus \{0\}$. *Then there exists $c \in \mathbb{C} \setminus \{0\}$ such that*

$$f(z) = c\, z^n \qquad (z \in A(r, R)).$$

If $n > 0$ then f preserves the two boundary components in the sense that $|z| \downarrow r$ forces $|f(z)| \downarrow r'$ and $|z| \uparrow R$ forces $|f(z)| \uparrow R'$. If $n < 0$ then composing the target with inversion $w \mapsto 1/w$ converts n to $|n|$.

Proof. Theorem 7.7 gives the explicit form $f(z) = cz^n$ and shows that properness forces the radii constraints $r' = |c|r^{|n|}$ and $R' = |c|R^{|n|}$. When $n > 0$ the map $|f(z)| = |c|\,|z|^n$ is strictly increasing in $|z|$, so approaching r forces approaching r' and approaching R forces approaching R'. When $n < 0$ the modulus scaling is the same, and composing with inversion switches the orientation class in π_1 and therefore flips the sign of the degree, as discussed in Proposition 7.6. ☐

Remark 7.10 (Incommensurable moduli prevent proper maps)*.* If $\operatorname{mod}(A')/\operatorname{mod}(A) \notin$, then no proper holomorphic map $A \to A'$ exists. In particular, there is no way to "wrap" one annulus onto another with a non-integer modulus ratio. This is one of the cleanest ways modulus acts as a mapping constraint: it is not merely geometric, it is arithmetic. The same phenomenon appears in the classical theory of coverings of doubly connected Riemann surfaces [2, 4].

7.4. **Non-proper maps and homotopy collapse.** Proper maps are rigid. If we drop properness, holomorphic maps $A \to A'$ still carry a sharp π_1-invariant, but they are no longer forced to scale modulus. In the strip model, the difference is exactly whether the lift shifts by a nonzero multiple of $2\pi i$.

Proposition 7.11 (Lift periodicity encodes the induced map on π_1)**.** *Let $A = A(r, R)$ and $A' = A(r', R')$ be round annuli. Let $f \in \operatorname{Hol}(A, A')$ and let S, S' be the associated logarithmic strips. There exists a holomorphic lift $F : S \to S'$ such that*

$$\exp(F(w)) = f(e^w) \qquad (w \in S).$$

For any such lift there is an integer $n \in \mathbb{Z}$ with

$$F(w + 2\pi i) = F(w) + 2\pi i\, n \qquad (w \in S),$$

and this integer equals $\deg(f)$ *as defined in Definition 7.3.*

Proof. Because $\exp : S' \to A'$ is a covering map and S is simply connected, the composition $f \circ \exp : S \to A'$ has a holomorphic lift $F : S \to S'$ with $\exp \circ F = f \circ \exp$. For any $w \in S$ one has

$$\exp(F(w + 2\pi i)) = f(e^{w+2\pi i}) = f(e^w) = \exp(F(w)),$$

so $F(w + 2\pi i) - F(w) \in 2\pi i\,\mathbb{Z}$. The function

$$n(w) := \frac{F(w + 2\pi i) - F(w)}{2\pi i}$$

is holomorphic on S and takes values in the discrete set $\mathbb{Z}$, hence is constant because S is connected. Call the constant n.

To identify n with $\deg(f)$, take the generator loop in A given by $t \mapsto e^{x+it}$ for fixed $x \in (\log r, \log R)$. Upstairs, increasing t by 2π corresponds to translating w by $2\pi i$. The shift $F(w + 2\pi i) = F(w) + 2\pi i n$ says the image loop winds n times around the target hole. This is exactly the induced map on $\pi_1(A) \cong \mathbb{Z} \to \pi_1(A') \cong \mathbb{Z}$ and therefore equals $\deg(f)$ [3, 4]. □

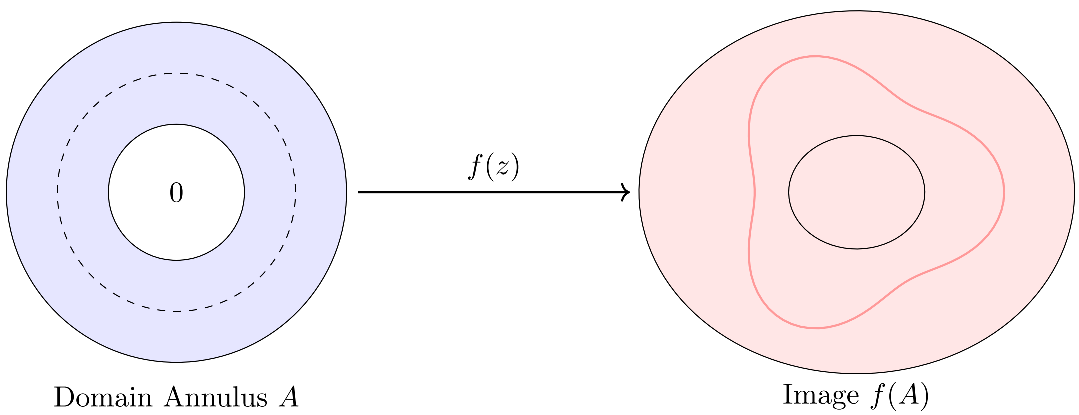


Figure 19. The transformation of a circular annulus under a holomorphic map. The modulus $M(A)$ restricts how much the image can "twist" or "pinch".

Lemma 7.12 (Degree zero is equivalent to a holomorphic logarithm on the annulus)**.** *Let A be an annulus and let $A' = A(r', R')$ be a round annulus. For $f \in \mathrm{Hol}(A, A')$, the following are equivalent. The degree satisfies $\deg(f) = 0$. There exists a holomorphic function $g \in \mathrm{Hol}(A, \mathbb{C})$ such that*

$$f(z) = e^{g(z)} \qquad (z \in A),$$

so g is a single-valued holomorphic logarithm of f on A.

Proof. Assume first that $\deg(f) = 0$. Choose a round model for A and lift f to $F : S \to S'$ as in Proposition 7.11. The degree condition is exactly $F(w + 2\pi i) = F(w)$, so F is $2\pi i$-periodic. Define g on A by

$$g(z) := F(\log z),$$

where $\log z$ means any logarithm $w \in S$ with $e^w = z$. This is well-defined because F is periodic. Holomorphy follows from local inversion of exp exactly as in Proposition 4.7. By construction,

$$e^{g(z)} = e^{F(\log z)} = f(e^{\log z}) = f(z),$$

so $f = e^g$.

Conversely, if $f = e^g$ for some holomorphic g on A, then f has a single-valued logarithm and its image loop has zero net change in argument on any generator loop. Equivalently, the winding number of $f \circ \gamma$ around 0 is 0, hence $\deg(f) = 0$ by Lemma 7.4. □

Proposition 7.13 (Homotopy dichotomy and collapse). *Let $f : A \to A'$ be holomorphic between annuli. Either $\deg(f) = n \neq 0$, in which case f wraps a generator loop around the target hole $|n|$ times, or $\deg(f) = 0$, in which case f is null-homotopic and admits a holomorphic logarithm after identifying the target with a round annulus. If f is proper, then $\deg(f) \neq 0$.*

Proof. The induced map on π_1 is a homomorphism $\mathbb{Z} \to \mathbb{Z}$, hence multiplication by an integer n. This produces the dichotomy $\deg(f) = 0$ or $\deg(f) \neq 0$. Proposition 7.11 shows the strip-lift identity $F(w + 2\pi i) = F(w) + 2\pi i n$ and therefore makes the wrapping interpretation literal in logarithmic coordinates. Lemma 7.12 identifies the case $n = 0$ with the existence of a holomorphic logarithm, which is the analytic form of "homotopy collapse".

If f is proper and nonconstant, Lemma 7.5 shows it is a finite-sheeted covering away from critical points and therefore induces an injective map on π_1, which forces $n \neq 0$. □

Remark 7.14 (How modulus re-enters for non-proper maps). For non-proper maps, modulus no longer imposes the discrete scaling constraint $\mathrm{mod}(A') = n\mathrm{mod}(A)$. Instead, modulus controls the quantitative behavior through the two analytic mechanisms already developed. Large modulus means wide strip width $W = 2\pi\mathrm{mod}(A)$, which forces exponential suppression of nonzero Fourier modes in strip lifts (Section 4), and it also produces a very short core geodesic and strong hyperbolic contraction structure (Section 5 and Section 6). Small modulus makes the hyperbolic density scale like $1/\mathrm{mod}(A)$ even in the core, which is exactly the scale at which the constants in Schwarz–Pick derivative bounds blow up. In this sense, modulus remains the quantitative complexity parameter even when it ceases to be a rigid mapping constraint.

8. Compactness, normal families, and degeneration regimes controlled by modulus

8.1. Montel normality for uniformly bounded families.

The first compactness principle is classical: uniformly bounded holomorphic families are normal. We record it in a form that is compatible with the annulus viewpoint developed earlier, because on annuli we will often combine Montel with the modulus-dependent coefficient suppression from Section 4. Standard references for normal families and Montel's theorem include [3, 8].

Definition 8.1 (Normal family). Let $\Omega \subset \mathbb{C}$ be a domain. A family $\mathcal{F} \subset \mathrm{Hol}(\Omega, \mathbb{C})$ is called *normal* if every sequence $(f_k) \subset \mathcal{F}$ admits a subsequence (f_{k_j}) that converges uniformly on compact subsets of Ω either to a holomorphic function $f \in \mathrm{Hol}(\Omega, \mathbb{C})$ or to the constant function ∞ in the spherical metric on $\widehat{\mathbb{C}}$.

Theorem 8.2 (Montel). *Let $\Omega \subset \mathbb{C}$ be a domain. Assume $\mathcal{F} \subset \mathrm{Hol}(\Omega, \mathbb{C})$ is locally uniformly bounded in the sense that for every compact set $K \subset \Omega$ there exists $M_K < \infty$ such that*

$$\sup_{f \in \mathcal{F}} \sup_{z \in K} |f(z)| \leq M_K.$$

Then $\mathcal{F}$ is normal. In particular, if there exists $M < \infty$ with $\sup_{f \in \mathcal{F}} \sup_{z \in \Omega} |f(z)| \leq M$, then $\mathcal{F}$ is normal. [3, 8]

Proof. Fix a compact set $K \subset \Omega$. Define the distance from K to the boundary by

$$\delta := \mathrm{dist}(K, \partial\Omega) = \inf\{|z - \zeta| : z \in K,\ \zeta \in \partial\Omega\}.$$

This number is strictly positive because K is compact, $\partial\Omega$ is closed, and $K \cap \partial\Omega = \varnothing$. Indeed, if $\delta = 0$ then there exist $z_n \in K$ and $\zeta_n \in \partial\Omega$ with $|z_n - \zeta_n| \to 0$; by compactness a subsequence

$z_{n_j} \to z_\infty \in K$, and then $|\zeta_{n_j} - z_\infty| \le |\zeta_{n_j} - z_{n_j}| + |z_{n_j} - z_\infty| \to 0$, so $\zeta_{n_j} \to z_\infty \in \partial\Omega$ because $\partial\Omega$ is closed, contradicting $z_\infty \in K \subset \Omega$.

Consider the closed $\delta/2$-neighborhood of K inside Ω,

$$K_{\delta/2} := \{z \in \Omega : \operatorname{dist}(z, K) \le \delta/2\}.$$

This set is compact and contained in Ω. Compactness follows because it is closed and bounded in $\mathbb{C}$. Containment in Ω follows because if $z \in K_{\delta/2}$ there exists $w \in K$ with $|z - w| \le \delta/2$, and for any $\zeta \in \partial\Omega$ one has $|w - \zeta| \ge \delta$ by definition of δ, hence

$$|z - \zeta| \ge |w - \zeta| - |z - w| \ge \delta - \delta/2 = \delta/2 > 0,$$

so $z \notin \partial\Omega$.

By the local uniform boundedness assumption applied to the compact set $K_{\delta/2}$, there exists $M < \infty$ such that $|f(z)| \le M$ for all $f \in \mathcal{F}$ and all $z \in K_{\delta/2}$.

Now fix $z \in K$ and apply Cauchy's integral formula for derivatives on the circle $|\zeta - z| = \delta/2$, which is contained in $K_{\delta/2}$:

$$f'(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = \delta/2} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Taking absolute values and using $|f(\zeta)| \le M$ and $|\zeta - z| = \delta/2$ yields

$$|f'(z)| \le \frac{1}{2\pi} \int_{|\zeta - z| = \delta/2} \frac{M}{(\delta/2)^2}\, |d\zeta| = \frac{1}{2\pi} \cdot \frac{M}{(\delta/2)^2} \cdot 2\pi \frac{\delta}{2} = \frac{2M}{\delta}.$$

Since $z \in K$ was arbitrary, this gives the uniform derivative bound

$$\sup_{f \in \mathcal{F}} \sup_{z \in K} |f'(z)| \le \frac{2M}{\delta}.$$

For any $z_1, z_2 \in K$, integrate f' along the line segment $\sigma(t) = z_1 + t(z_2 - z_1)$ to obtain

$$f(z_2) - f(z_1) = \int_0^1 f'(\sigma(t))\,(z_2 - z_1)\, dt, \qquad |f(z_2) - f(z_1)| \le \frac{2M}{\delta} |z_2 - z_1|.$$

Thus $\mathcal{F}$ is equi-Lipschitz on K, hence equicontinuous on K, and it is uniformly bounded on K because $K \subset K_{\delta/2}$.

By Arzelà–Ascoli, every sequence in $\mathcal{F}$ has a subsequence converging uniformly on K to a continuous limit. Choose an exhaustion $K_1 \subset K_2 \subset \cdots$ of Ω by compact sets with union Ω; one convenient choice is

$$K_j = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \ge 1/j,\ |z| \le j\}.$$

Diagonal extraction yields a subsequence converging uniformly on each K_j, hence uniformly on compact subsets of Ω.

Finally, the limit is holomorphic by passage to the limit in Cauchy integrals on small disks: if $\overline{D(z_0, r_0)} \subset \Omega$ and $f_{k_j} \to f$ uniformly on that closed disk, then for $z \in D(z_0, r_0)$ one has

$$f_{k_j}(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r_0} \frac{f_{k_j}(\zeta)}{\zeta - z}\, d\zeta,$$

and letting $j \to \infty$ under the integral sign gives the same formula for f, proving holomorphy on the disk. Since z_0 is arbitrary, $f \in \operatorname{Hol}(\Omega, \mathbb{C})$, so $\mathcal{F}$ is normal. [3, 8] □

Remark 8.3 (Why modulus matters even for Montel). Montel's theorem is qualitative and makes no explicit mention of $\operatorname{mod}(A)$. On annuli, a common route to local uniform boundedness is to bound finitely many Laurent modes and then control the tail by the exponential suppression factor from Theorem 4.11. That suppression becomes strong when the strip width

$$W = \log(R/r) = 2\pi \operatorname{mod}(A)$$

is large, and it turns qualitative compactness into quantitative precompactness with explicit rates. [3, 12]

8.2. **Degeneration as** $\operatorname{mod}(A) \to \infty$ **(thin annuli).** We now study families on varying annuli A_k with $\operatorname{mod}(A_k) \to \infty$. In strip coordinates this is the long-cylinder regime: the width $W_k = 2\pi\operatorname{mod}(A_k)$ tends to $+\infty$. The key quantitative phenomenon is angular rigidity on the mid-cylinder, which is exactly the Fourier/Laurent decay mechanism from Section 4.

Definition 8.4 (Thin annuli normalization)**.** Let (m_k) be a sequence with $m_k \to \infty$. Define

$$A_k := A(q_k, 1), \qquad q_k := e^{-2\pi m_k}.$$

Then $\operatorname{mod}(A_k) = m_k$ and the covering strip has width $W_k = \log(1/q_k) = 2\pi m_k \to \infty$.

Theorem 8.5 (Angular rigidity on thin annuli)**.** *Let $A_k = A(q_k, 1)$ with $q_k = e^{-2\pi m_k}$ and $m_k \to \infty$. Let $f_k \in \operatorname{Hol}(A_k, \mathbb{D})$ and write the Laurent series*

$$f_k(z) = \sum_{n\in\mathbb{Z}} a_n^{(k)} z^n.$$

Then every nonzero mode is exponentially suppressed:

$$|a_n^{(k)}| \le q_k^{|n|/2} = \exp(-\pi|n|\, m_k) \qquad (n \in \mathbb{Z}).$$

On the core circle $|z| = \sqrt{q_k}$ the function is uniformly close to its constant mode:

$$\sup_{|z|=\sqrt{q_k}} \big|f_k(z) - a_0^{(k)}\big| \le \frac{2q_k}{1-q_k}.$$

Consequently, after passing to a subsequence, the constants $a_0^{(k)} \in \overline{\mathbb{D}}$ converge and the maps converge uniformly on the corresponding core circles to a constant. [3, 12]

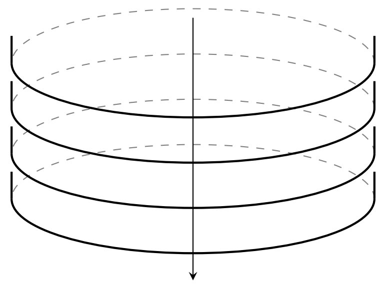
Proof. Since each f_k is disk-valued, $\|f_k\|_{L^\infty(A_k)} \le 1$. The coefficient estimate $|a_n^{(k)}| \le q_k^{|n|/2}$ is precisely Theorem 4.11 in the normalized case (with $M = 1$). For $|z| = \sqrt{q_k}$ one has

$$f_k(z) - a_0^{(k)} = \sum_{n\neq 0} a_n^{(k)} z^n, \qquad |f_k(z) - a_0^{(k)}| \le \sum_{n\neq 0} |a_n^{(k)}|\,|z|^{|n|}.$$

Here $|z|^{|n|} = q_k^{|n|/2}$, so each term is bounded by $q_k^{|n|/2} \cdot q_k^{|n|/2} = q_k^{|n|}$. Summing yields

$$|f_k(z) - a_0^{(k)}| \le \sum_{n\neq 0} q_k^{|n|} = 2\sum_{n\ge 1} q_k^n = \frac{2q_k}{1-q_k},$$

and taking the supremum gives the stated bound. Since $|a_0^{(k)}| \le 1$, compactness of $\overline{\mathbb{D}}$ gives a convergent subsequence; the uniform core-circle bound then forces convergence of f_k to the same constant on those core circles. [3] □



To pass from "core-circle convergence" to convergence on fixed planar domains, one simply restricts away from the shrinking inner boundary and then applies Montel on a fixed annulus.

Theorem 8.6 (Fixed-domain normality in the thin regime)**.** *Let $A_k = A(q_k, 1)$ with $q_k \to 0$ and $f_k \in \mathrm{Hol}(A_k, \mathbb{D})$. Fix $\alpha \in (0,1)$ and consider the fixed annulus $A^{(\alpha)} = \{z \in \mathbb{C} : \alpha < |z| < 1\}$. For all sufficiently large k one has $A^{(\alpha)} \subset A_k$, and the restricted family $\{f_k|_{A^{(\alpha)}}\}$ is normal on $A^{(\alpha)}$. Hence there exists a subsequence converging uniformly on compact subsets of $A^{(\alpha)}$ to a holomorphic limit with values in $\overline{\mathbb{D}}$.* [3]

Proof. Uniform boundedness $|f_k| \le 1$ holds on A_k, hence on $A^{(\alpha)}$ for all large k. Montel's theorem then gives normality on the fixed domain. [3] □

Remark 8.7 (Punctured-disk degeneration). Letting $\alpha \downarrow 0$ through a sequence and diagonalizing yields subsequential convergence on every compact subset of $\mathbb{D}^*$. If one additionally has a uniform bound near 0, then the limit extends across 0 by the removable singularity theorem, so thin annuli naturally degenerate toward punctured-disk objects. [12, 8]

8.3. **Degeneration as** $\mathrm{mod}(A) \to 0$ **(thick annuli).** In the opposite regime $\mathrm{mod}(A_k) \to 0$, the strip width $W_k = 2\pi \mathrm{mod}(A_k) \to 0$ and the cylinder becomes short. The hyperbolic density has an explicit prefactor π/W_k, so the intrinsic geometry becomes very large in Euclidean units, and Schwarz–Pick derivatives blow up at the same scale.

Definition 8.8 (Thick annuli model)**.** Let $m_k \downarrow 0$ and set $W_k = 2\pi m_k \downarrow 0$. Choose annuli $A_k = A(r_k, R_k)$ such that $\log(R_k/r_k) = W_k$, equivalently $R_k/r_k = e^{W_k} \to 1$.

Proposition 8.9 (Hyperbolic density scale on the core)**.** *Let $A = A(r, R)$ with $W = \log(R/r)$. On the core circle $|z| = \sqrt{rR}$ the hyperbolic density satisfies*

$$\rho_A(z) = \frac{\pi}{W} \cdot \frac{1}{|z|}.$$

Thus for a fixed Euclidean scale one has $\rho_A \asymp 1/W \asymp 1/\mathrm{mod}(A)$ as $\mathrm{mod}(A) \to 0$. [2]

Proof. On $|z| = \sqrt{rR}$ one has $\log(|z|/r) = W/2$. In Theorem 5.8 the trigonometric factor becomes $\csc(\pi/2) = 1$, yielding the identity. [2] □

Corollary 8.10 (Derivative bounds blow up like $1/\mathrm{mod}(A)$ in the thick regime)**.** *Let $A = A(r, R)$ with $W = \log(R/r)$ and let $f \in \mathrm{Hol}(A, \mathbb{D})$. On the core circle $|z| = \sqrt{rR}$ one has*

$$|f'(z)| \le \frac{1}{2}\rho_A(z) = \frac{1}{2} \cdot \frac{\pi}{W} \cdot \frac{1}{|z|}.$$

Hence the bound grows like $1/W \sim 1/\mathrm{mod}(A)$ as $\mathrm{mod}(A) \to 0$. [12, 3]

Proof. Schwarz–Pick gives $|f'(z)| \le \rho_A(z)/\rho_{\mathbb{D}}(f(z))$ and $\rho_{\mathbb{D}}(\zeta) \ge 2$ on $\mathbb{D}$, so $|f'(z)| \le \rho_A(z)/2$. Insert Proposition 8.9. [3] □

Remark 8.11 (What changes as $\operatorname{mod}(A) \to 0$). The annulus becomes short in logarithmic scale, but the hyperbolic metric must remain complete with curvature -1. The only way to achieve this is to increase the density. This increase appears as π/W in the explicit formula for ρ_A and propagates into every Schwarz–Pick bound. [2]

8.4. **Application: quantitative removability thresholds.** The thin-annulus bounds can be read as a quantitative version of the removable singularities theorem. The principal part coefficients decay at an exponential rate governed by the modulus.

Theorem 8.12 (Quantitative extension across a puncture)**.** *Let f be holomorphic on $\mathbb{D}^* = \{0 < |z| < 1\}$ and satisfy $|f(z)| \le M$. Fix $q \in (0,1)$ and restrict f to $A(q,1)$. If*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \qquad \text{on } A(q,1),$$

then for every $n \ge 1$,

$$|a_{-n}| \le Mq^n.$$

Equivalently,

$$|a_{-n}| \le M \exp\big(-2\pi n \operatorname{mod}(A(q,1))\big).$$

In particular, f extends holomorphically to $\mathbb{D}$. [8]

Proof. For $n \ge 1$ and any $\rho \in (q,1)$, Cauchy's formula for Laurent coefficients gives

$$a_{-n} = \frac{1}{2\pi i}\int_{|z|=\rho} f(z)\, z^{n-1}\, dz,$$

so

$$|a_{-n}| \le \frac{1}{2\pi}\int_{|z|=\rho} |f(z)|\,|z|^{n-1}\,|dz| \le \frac{1}{2\pi}\cdot M \cdot \rho^{n-1}\cdot 2\pi\rho = M\rho^n.$$

Letting $\rho \downarrow q$ yields $|a_{-n}| \le Mq^n$. The exponential reparametrization is the identity $q = \exp(-2\pi \operatorname{mod}(A(q,1)))$. [8, 12] ☐

Quantity	**Euclidean Formula** ($z \in A$)	**Cylinder Form** ($t = \log|z|$)	**Asymptotics** ($W \to \infty$)
Conf. Width	$\log(R/r)$	$W = 2\pi \cdot \operatorname{Mod}(A)$	Scaling factor
Outer Measure	$\omega_R(z) = \dfrac{\log(|z|/r)}{\log(R/r)}$	$\omega_R(t) = \dfrac{t - \log r}{W}$	$\sim t/W$ (Linear)
Inner Measure	$\omega_r(z) = \dfrac{\log(R/|z|)}{\log(R/r)}$	$\omega_r(t) = 1 - \dfrac{t-\log r}{W}$	$\sim 1 - t/W$
Mode Damping	$h_n(z)$ s.t. $h|_{\partial A} = (e^{in\theta}, 0)$	$\dfrac{\sinh(n(t-\log r))}{\sinh(nW)}$	$\sim e^{-n(W-\tau)}$
Hadamard	$|f(z)| \le M_r^{\omega_r(z)} M_R^{\omega_R(z)}$	$\log|f| \le \left(1 - \frac{\tau}{W}\right)\ln M_r + \frac{\tau}{W}\ln M_R$	Convexity
Hyp. Density	$\rho_A(z) = \frac{\pi}{2|z|\log(R/r)}\csc\left(\pi\frac{\log(|z|/r)}{\log(R/r)}\right)$	$\rho_A(t) = \dfrac{\pi}{2W}\csc\left(\frac{\pi\tau}{W}\right)$	$\asymp \frac{1}{\operatorname{dist}(t,\partial)}$

Table 2. Detailed quantitative dictionary for the annulus $A(r,R)$. The modulus enters as the cylinder width $W = 2\pi\operatorname{Mod}(A)$. The variable $t = \log|z|$ represents the logarithmic radial coordinate.

Remark 8.13 (Modulus as an exponential rate parameter). Since $\operatorname{mod}(A(q,1)) = \frac{1}{2\pi}\log(1/q)$, the coefficient decay q^n becomes

$$q^n = \exp\big(-2\pi n \operatorname{mod}(A(q,1))\big).$$

This is the same exponential mechanism as Fourier decay on wide strips, now repackaged as a boundary-removability estimate. In particular, “large modulus” can be read as “strong suppression of principal parts” with an explicit rate. [12]

9. Boundary behavior, harmonic measure, and quantitative two-boundary interaction

9.1. **Harmonic measure on an annulus.** The annulus has two boundary components, so boundary value problems naturally split into “inner-boundary data” and “outer-boundary data”. The quantitative interaction between these two pieces of data is controlled by the modulus, and the cleanest object that measures this interaction is harmonic measure. Standard references for harmonic measure, Perron solutions, and Poisson representations include [12, 27, 3].

Definition 9.1 (Harmonic measure)**.** Let $\Omega \subset \mathbb{C}$ be a domain and let $E \subset \partial\Omega$ be a Borel subset of the boundary. The *harmonic measure of E as seen from $z \in \Omega$*, denoted $\omega(z, E, \Omega)$, is defined as follows: $\omega(\cdot, E, \Omega)$ is the unique harmonic function $u : \Omega \to [0, 1]$ such that

$$u|_E = 1, \qquad u|_{\partial\Omega\setminus E} = 0,$$

in the appropriate sense of boundary limits (classical if $\partial\Omega$ is smooth, otherwise in the Perron sense). Then

$$\omega(z, E, \Omega) := u(z).$$

Remark 9.2 (Probabilistic interpretation)*.* If (B_t) is planar Brownian motion started at $z \in \Omega$ and $\tau = \inf\{t : B_t \notin \Omega\}$ is the exit time, then

$$\omega(z, E, \Omega) = \mathbb{P}_z(B_\tau \in E).$$

We will not use probability in the proofs, but it explains why harmonic measure is the correct “boundary influence” quantity. [27]

We now compute harmonic measure explicitly for the round annulus.

Theorem 9.3 (Explicit harmonic measure for the boundary circles)**.** *Let $A = A(r, R)$. Let $C_r = \{|z| = r\}$ and $C_R = \{|z| = R\}$ be the inner and outer boundary circles. Define $u(z) := \omega(z, C_R, A)$, the harmonic measure of the outer boundary as seen from z. Then*

$$u(z) = \frac{\log|z| - \log r}{\log R - \log r} = \frac{\log(|z|/r)}{\log(R/r)}.$$

Similarly,

$$\omega(z, C_r, A) = 1 - u(z) = \frac{\log R - \log|z|}{\log R - \log r} = \frac{\log(R/|z|)}{\log(R/r)}.$$

In particular, u depends on the annulus only through the width

$$W = \log(R/r) = 2\pi \operatorname{mod}(A),$$

and it is affine in the logarithmic radius $\log|z|$*.* [12, 27]

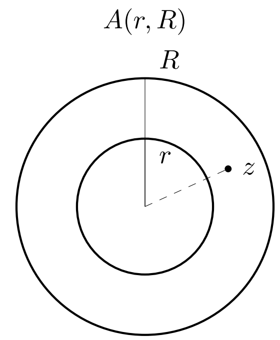
Proof. Set $W := \log R - \log r = \log(R/r)$ and define the candidate function

$$u(z) := \frac{\log|z| - \log r}{W}.$$

If $|z| = r$, then $\log|z| = \log r$ and $u(z) = 0$, while if $|z| = R$, then $\log|z| = \log R$ and $u(z) = 1$.

It remains to check $\Delta u = 0$ on A. Since u is a constant multiple of $\log|z|$, it suffices to show $\log|z|$ is harmonic on $\mathbb{C} \setminus \{0\}$. Write $z = x + iy$, so $|z| = \sqrt{x^2 + y^2}$ and $\log|z| = \frac{1}{2}\log(x^2 + y^2)$. Then

$$\partial_x(\log|z|) = \frac{x}{x^2 + y^2}, \qquad \partial_y(\log|z|) = \frac{y}{x^2 + y^2},$$



$$\omega(z, C_R, A) = \frac{\log(|z|/r)}{\log(R/r)} = \frac{\log|z| - \log r}{W}$$

Figure 20. For a round annulus, the harmonic measure of the outer boundary is exactly the normalized logarithmic radius. The modulus enters only through $W = \log(R/r) = 2\pi \mathrm{mod}(A)$.

and

$$\partial_{xx}(\log|z|) = \frac{y^2 - x^2}{(x^2+y^2)^2}, \qquad \partial_{yy}(\log|z|) = \frac{x^2 - y^2}{(x^2+y^2)^2}.$$

Adding gives $\Delta(\log|z|) = 0$, hence u is harmonic on A.

If v is any harmonic function on A with the same boundary values, then $v - u$ is harmonic with zero boundary values on both boundary circles, so $v - u \equiv 0$ by the maximum principle. Therefore u is the unique harmonic function with the stated boundary values, hence it equals $\omega(\cdot, C_R, A)$ by Definition 9.1. [12, 3] □

Remark 9.4 (Quantitative two-boundary interaction in one line). If $|z| = re^t$ with $t \in (0, W)$, then

$$\omega(z, C_R, A) = \frac{t}{W} = \frac{t}{2\pi \,\mathrm{mod}(A)}.$$

Thus moving a fixed logarithmic distance t away from the inner boundary becomes less significant as $\mathrm{mod}(A)$ grows, because the cylinder is longer and the normalized position t/W becomes smaller. [12]

9.2. **Poisson kernels for the annulus and explicit integral formulas.** We now write explicit integral formulas for harmonic functions on an annulus in terms of boundary data. This is the precise analytic mechanism by which boundary values on one circle influence the interior and the other circle. The annulus Dirichlet problem can be solved by separation of variables in logarithmic coordinates; see [27, 12].

Definition 9.5 (Boundary data and harmonic extension problem)**.** Let $A = A(r, R)$. Given two continuous 2π-periodic functions

$$\varphi_r : S^1 \to \mathbb{R}, \qquad \varphi_R : S^1 \to \mathbb{R},$$

we seek a harmonic function u on A such that

$$u(re^{i\theta}) = \varphi_r(e^{i\theta}), \qquad u(Re^{i\theta}) = \varphi_R(e^{i\theta}).$$

Remark 9.6 (Strip coordinates and separability). Write $z = e^w$ with $w = x + iy$, so $x = \Re w = \log|z|$ and $y = \Im w = \arg z$. Then the annulus becomes the cylinder

$$(x, y) \in (\log r, \log R) \times (\mathbb{R}/2\pi\mathbb{Z}),$$

and harmonicity becomes the Laplace equation in (x, y) with periodicity in y. Fourier series in y diagonalize the operator, turning the problem into a family of one-dimensional ODEs in x. [27]

Theorem 9.7 (Fourier-series Poisson formula for the annulus)**.** *Let $A = A(r, R)$ and let $W = \log(R/r)$. Assume φ_r, φ_R have Fourier series*

$$\varphi_r(\theta) = \sum_{n\in\mathbb{Z}} \widehat{\varphi_r}(n)e^{in\theta}, \qquad \varphi_R(\theta) = \sum_{n\in\mathbb{Z}} \widehat{\varphi_R}(n)e^{in\theta},$$

with

$$\widehat{\varphi}(n) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta)e^{-in\theta}\, d\theta.$$

Then the unique harmonic function u on A with these boundary values is

$$u(\rho e^{i\theta}) = \widehat{\varphi_r}(0)\cdot\frac{\log(R/\rho)}{W} + \widehat{\varphi_R}(0)\cdot\frac{\log(\rho/r)}{W} + \sum_{n\neq 0}\Big(A_n(\rho)\widehat{\varphi_r}(n) + B_n(\rho)\widehat{\varphi_R}(n)\Big)e^{in\theta},$$

where for $n \neq 0$,

$$A_n(\rho) := \frac{\sinh\Big(|n|\log(R/\rho)\Big)}{\sinh(|n|W)}, \qquad B_n(\rho) := \frac{\sinh\Big(|n|\log(\rho/r)\Big)}{\sinh(|n|W)}.$$

In particular, for each fixed interior radius ρ, nonzero Fourier modes transmitted from either boundary are suppressed by the denominator $\sinh(|n|W)$*, which decays exponentially in $W = 2\pi\mathrm{mod}(A)$ for large modulus.* [27]

Proof. Define $v(x, y) := u(e^{x+iy})$ on $(\log r, \log R)\times(\mathbb{R}/2\pi\mathbb{Z})$. Then v is harmonic and 2π-periodic in y, hence admits a Fourier expansion

$$v(x, y) = \sum_{n\in\mathbb{Z}} v_n(x)e^{iny}.$$

The Laplace equation $v_{xx} + v_{yy} = 0$ implies $v_n''(x) - n^2 v_n(x) = 0$ for each n.

For $n = 0$ one has $v_0''(x) = 0$, so $v_0(x) = \alpha x + \beta$. Imposing $v_0(\log r) = \widehat{\varphi_r}(0)$ and $v_0(\log R) = \widehat{\varphi_R}(0)$ yields

$$v_0(x) = \widehat{\varphi_r}(0)\frac{\log R - x}{W} + \widehat{\varphi_R}(0)\frac{x - \log r}{W}.$$

For $n \neq 0$, write $k = |n| \geq 1$. The ODE $v_n'' - k^2 v_n = 0$ has general solution $v_n(x) = C_n \cosh(k(x - \log r)) + D_n \sinh(k(x - \log r))$. The boundary condition at $x = \log r$ gives $C_n = \widehat{\varphi_r}(n)$, and the boundary condition at $x = \log R = \log r + W$ determines D_n. Rewriting the resulting expression in terms of $t = x - \log r = \log(\rho/r)$ gives exactly the stated coefficients $A_n(\rho)$ and $B_n(\rho)$.

Returning to $u(\rho e^{i\theta}) = v(\log\rho, \theta)$ completes the formula. Uniqueness follows from the maximum principle for the annulus Dirichlet problem. [27, 12] □

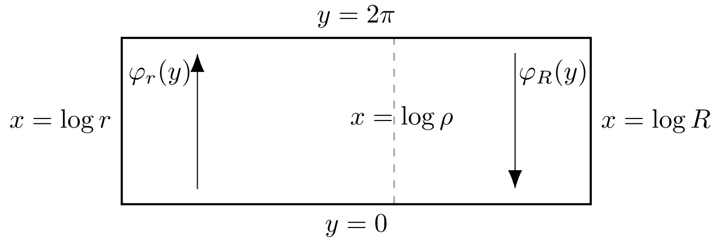
Remark 9.8 (Exponential damping across the cylinder). Fix $\rho \in (r, R)$. For $n \neq 0$, the factors

$$A_n(\rho) = \frac{\sinh(|n|\log(R/\rho))}{\sinh(|n|W)}, \qquad B_n(\rho) = \frac{\sinh(|n|\log(\rho/r))}{\sinh(|n|W)}$$

satisfy $|A_n(\rho)| + |B_n(\rho)| \leq \cosh(|n|W)/\sinh(|n|W) = \coth(|n|W)$, and in particular

$$|A_n(\rho)| + |B_n(\rho)| \lesssim e^{-|n|W} \qquad \text{as } W \to \infty.$$

This is the harmonic analogue of Fourier/Laurent decay for holomorphic functions on wide strips. [12]



mode n transmission is weighted by $\sinh(|n|W)^{-1}$, $\quad W = \log(R/r) = 2\pi \mathrm{mod}(A)$

Figure 21. Annulus Dirichlet data becomes a periodic strip problem. Fourier separation in y produces explicit mode-transfer factors $A_n(\rho)$ and $B_n(\rho)$, whose denominators $\sinh(|n|W)$ encode modulus-controlled damping.

9.3. **Holomorphic maps with boundary constraints.** We now translate the harmonic-measure and Poisson formulas into quantitative statements for holomorphic functions. The basic device is that $\log |f|$ is harmonic away from zeros, and subharmonic in general, so boundary control of $|f|$ propagates into the interior with weights given by harmonic measure. [8, 27]

Proposition 9.9 (Two-boundary interpolation for nonvanishing holomorphic functions)**.** *Let $A = A(r, R)$ and let $f \in \mathrm{Hol}(A, \mathbb{C})$ be nonvanishing. Define*

$$M_r := \sup_{|z|=r} |f(z)|, \qquad M_R := \sup_{|z|=R} |f(z)|.$$

Then for every $\rho \in (r, R)$ and every θ,

$$|f(\rho e^{i\theta})| \le M_r^{\omega(\rho e^{i\theta}, C_r, A)} \, M_R^{\omega(\rho e^{i\theta}, C_R, A)} = M_r^{\frac{\log(R/\rho)}{W}} \, M_R^{\frac{\log(\rho/r)}{W}}, \qquad W = \log(R/r).$$

Equivalently,

$$\log |f(\rho e^{i\theta})| \le \frac{\log(R/\rho)}{W} \log M_r + \frac{\log(\rho/r)}{W} \log M_R.$$

Proof. Because f is holomorphic and nonvanishing, $\log |f|$ is harmonic on A. Let $h(z) = \log |f(z)|$. On $|z| = r$ one has $h \le \log M_r$, and on $|z| = R$ one has $h \le \log M_R$. Define the harmonic function

$$H(z) := \omega(z, C_r, A) \log M_r + \omega(z, C_R, A) \log M_R.$$

Then $H = \log M_r$ on C_r and $H = \log M_R$ on C_R by the defining boundary values of harmonic measure. Thus $h - H$ is harmonic and ≤ 0 on both boundary circles, so $h \le H$ on A by the maximum principle. Insert the explicit harmonic measure from Theorem 9.3 and exponentiate. [27] □

Remark 9.10 (Three-circles in modulus language)*.* Proposition 9.9 is the Hadamard three-circles inequality in its sharp logarithmic form: the weights are affine in $\log \rho$ and normalized by $W = \log(R/r) = 2\pi \mathrm{mod}(A)$. All dependence on the annulus geometry is funneled through the modulus. [8]

Proposition 9.11 (Quantitative stability of logarithmic moduli)**.** *Let $A = A(r, R)$ and let $f, g \in \mathrm{Hol}(A, \mathbb{C})$ be nonvanishing. Assume*

$$\sup_{|z|=R} \big|\log |f(z)| - \log |g(z)|\big| \le \varepsilon, \qquad \sup_{|z|=r} \big|\log |f(z)| - \log |g(z)|\big| \le \varepsilon.$$

Then for every $z \in A$ one has

$$\big|\log |f(z)| - \log |g(z)|\big| \le \varepsilon.$$

Quantity	Formula on $A(r,R)$	Modulus dependence
Outer harmonic measure	$\omega(z,C_R,A)=\dfrac{\log(\lvert z\rvert/r)}{\log(R/r)}$	normalized by $W=2\pi\mathrm{mod}(A)$
Inner harmonic measure	$\omega(z,C_r,A)=\dfrac{\log(R/\lvert z\rvert)}{\log(R/r)}$	normalized by $W=2\pi\mathrm{mod}(A)$
Mode damping ($n\neq 0$)	$\sinh(\lvert n\rvert W)^{-1}$	$\sim e^{-\lvert n\rvert W}$ as $W\to\infty$
Two-boundary interpolation	$\lvert f(\rho e^{i\theta})\rvert\le M_r^{\log(R/\rho)/W}M_R^{\log(\rho/r)/W}$	weights depend only on ρ via t/W

Table 3. A quantitative two-boundary dictionary. Harmonic measure provides the exact weights and the modulus enters through $W=\log(R/r)=2\pi\mathrm{mod}(A)$.

More generally, if the discrepancy is controlled only on one boundary circle, then the interior discrepancy is controlled by a harmonic-measure weight depending on the normalized logarithmic position and hence on $\mathrm{mod}(A)$. [27]

Proof. Set $h(z)=\log\lvert f(z)\rvert-\log\lvert g(z)\rvert$. This is harmonic on A because it is the difference of two harmonic functions. The hypothesis gives $\lvert h\rvert\le\varepsilon$ on each boundary circle, hence $h\le\varepsilon$ and $-h\le\varepsilon$ on ∂A. By the maximum principle applied to h and to $-h$, one obtains $\lvert h\rvert\le\varepsilon$ throughout A. [3] □

Remark 9.12 (One-sided control and the modulus weight). If one only controls boundary data on C_R and has no information on C_r, harmonic measure still gives a sharp decomposition: for harmonic h one can write

$$h(z)=\omega(z,C_R,A)\,h|_{C_R}\;+\;\omega(z,C_r,A)\,h|_{C_r}$$

in the Perron/Poisson sense, and at the level of bounds this yields

$$\lvert h(z)\rvert\le\omega(z,C_R,A)\sup_{C_R}\lvert h\rvert+\omega(z,C_r,A)\sup_{C_r}\lvert h\rvert.$$

Thus the unknown inner-boundary contribution is multiplied by

$$\omega(z,C_r,A)=\frac{\log(R/\lvert z\rvert)}{W},\qquad W=\log(R/r)=2\pi\mathrm{mod}(A).$$

When W is large and z is near the outer boundary, this weight is small, so outer boundary data almost determines the interior. This is a precise quantitative form of boundary decoupling controlled by the modulus. [12]

10. Worked examples and a quantitative "dictionary"

10.1. **Canonical examples.** We collect explicit maps and computations that serve as calibration points for the quantitative statements proved earlier. The purpose is to keep a reusable toolbox of model maps, and to make it easy to see where various bounds are sharp (or nearly sharp). Throughout we work on the round annulus $A(r,R)$ and we set

$$W:=\log(R/r)=2\pi\,\mathrm{mod}(A(r,R)).$$

All dependence on the annulus in this section can be re-expressed in terms of W (or equivalently $\mathrm{mod}(A)$).

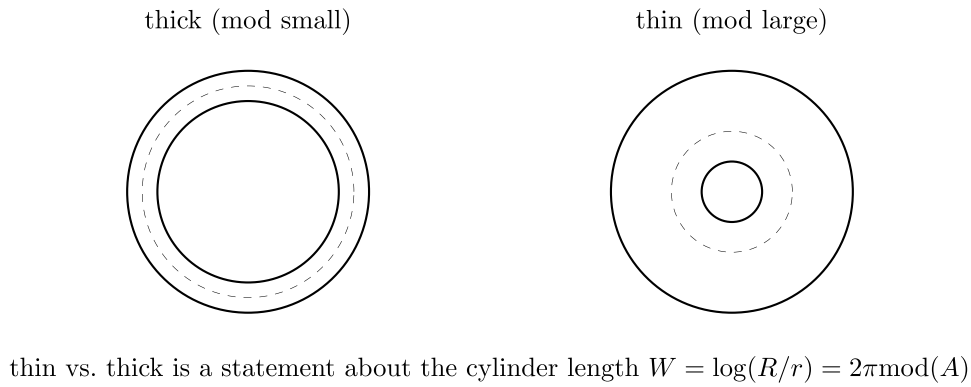


Figure 22. Side-by-side comparison: small modulus (thick annulus) versus large modulus (thin annulus).

10.1.1. *Power maps and modulus scaling.*

Example 10.1 (The power map $z \mapsto z^n$)**.** Fix $n \in$ and define $f(z) := z^n$. The map is holomorphic on $\mathbb{C} \setminus \{0\}$ and hence on $A(r, R)$. If $z \in A(r, R)$ then $r < |z| < R$, so $r^n < |z|^n < R^n$ and therefore $f(z) \in A(r^n, R^n)$, giving a holomorphic map

$$f : A(r, R) \to A(r^n, R^n).$$

This map is proper, and its degree is n because a positively oriented core circle winds n times around 0 under z^n. The modulus scaling is exact:

$$\mathrm{mod}(A(r^n, R^n)) = \frac{1}{2\pi} \log\left(\frac{R^n}{r^n}\right) = \frac{1}{2\pi} \cdot n \log\left(\frac{R}{r}\right) = n\,\mathrm{mod}(A(r, R)).$$

This is the concrete special case of the general scaling law for proper maps between annuli in Theorem 7.7. [2, 4, 3]

Example 10.2 (The scaled power map $z \mapsto cz^n$)**.** Fix $n \in$ and $c \in \mathbb{C} \setminus \{0\}$ and set $f(z) = cz^n$. Since $|f(z)| = |c|\,|z|^n$, one has

$$f(A(r, R)) = A(|c|r^n,\ |c|R^n).$$

Thus f is a proper map $A(r, R) \to A(r', R')$ precisely when $r' = |c|r^n$ and $R' = |c|R^n$, equivalently $R'/r' = (R/r)^n$ and $\mathrm{mod}(A') = n\mathrm{mod}(A)$. Up to a target rotation removing $\arg(c)$, this example is the general form of all proper maps between round annuli (Theorem 7.7). [2, 4]

Example 10.3 (Inversion and degree sign)**.** The inversion $I(z) = rR/z$ maps $A(r, R)$ biholomorphically to itself and swaps the two boundary circles. It reverses the preferred generator of $\pi_1(A)$, so it flips degree sign: $\deg(I \circ f) = -\deg(f)$. At the level of modulus, inversion changes nothing because $\mathrm{mod}(A(r, R))$ depends only on R/r and $R/r = (rR/r)/(rR/R)$. This is the simplest example where orientation on π_1 changes while the conformal type stays fixed. [3]

10.1.2. *Cylinder model and strip lifts.* The strip/cylinder model turns computations on annuli into linear algebra on rectangles. The basic input is the identity $W = \log(R/r) = 2\pi\mathrm{mod}(A)$, and the key output is that periodicity in the imaginary direction becomes Fourier series, which becomes Laurent series after exponentiation.

Example 10.4 (Lift of $z \mapsto z^n$ becomes affine in logarithmic target coordinates)**.** Let $f(z) = z^n$ on $A(r, R)$ and let $w = \log z$ in the covering strip $S = \{w : \log r < \Re w < \log R\}$. The strip-lift

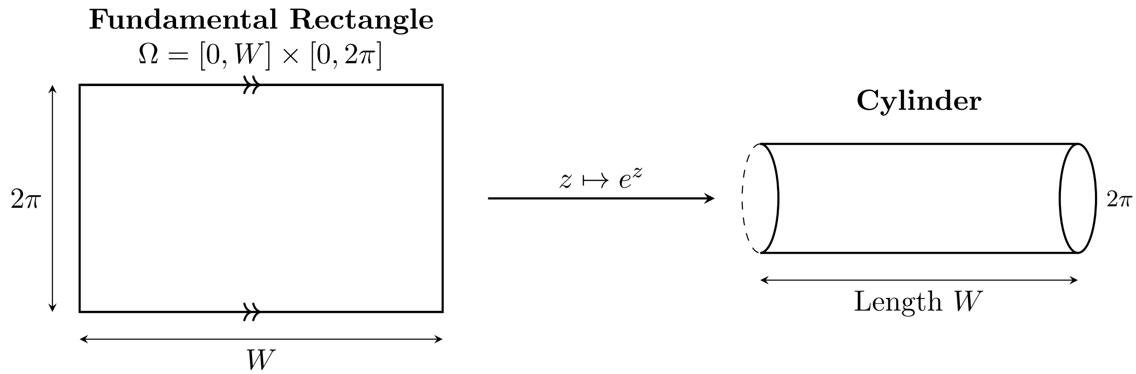


Figure 23. The modulus is exactly the Euclidean aspect ratio $W/(2\pi)$ of the fundamental rectangle. The rectangle is rolled up by identifying the edges of length W, creating a cylinder of length W and circumference 2π.

in the sense of Section 4 is $F(w) = f(e^w) = e^{nw}$. If we introduce logarithmic target coordinates $w' = \log(f(z))$ on the strip cover of the target annulus, then

$$w' = \log(e^{nw}) = nw$$

(modulo the deck translations), and the degree is encoded by the imaginary-period shift

$$w \mapsto w + 2\pi i \quad \Longrightarrow \quad w' \mapsto w' + 2\pi i\, n.$$

This is the clean model behind the general lift periodicity relation used in Section 7. [4, 3]

Example 10.5 (Pure Fourier modes on the strip are Laurent monomials)**.** Fix $n \in \mathbb{Z}$ and consider $G_n(w) := e^{nw}$ on the strip. The identity $e^{n(w+2\pi i)} = e^{nw}e^{2\pi in} = e^{nw}$ shows G_n is $2\pi i$-periodic precisely when $n \in \mathbb{Z}$. Under $z = e^w$ this gives $G_n(\log z) = z^n$. Thus Fourier expansion of a periodic holomorphic strip function is literally the Laurent expansion of the corresponding annulus function (Theorem 4.9). [3, 12]

Example 10.6 (A one-mode bounded calibration and exponential suppression)**.** On the centered strip $\{|\Re w| < W/2\}$, the single nonzero mode $F(w) = a_n e^{nw}$ has size

$$|F(w)| = |a_n| e^{n\Re w}.$$

If $n > 0$, the maximum over $|\Re w| \le W/2$ is $|a_n|e^{nW/2}$, while if $n < 0$, the maximum is $|a_n|e^{|n|W/2}$. Thus the condition $\sup_{|\Re w|<W/2} |F(w)| \le 1$ forces

$$|a_n| \le e^{-|n|W/2}.$$

This is exactly the shape of the general coefficient estimate (Theorem 4.11), and in the normalized annulus $A(q, 1)$ with $W = \log(1/q) = 2\pi\mathrm{mod}(A)$ it reads

$$|a_n| \le q^{|n|/2} = \exp(-\pi|n|\mathrm{mod}(A)).$$

This is the analytic mechanism behind angular rigidity on thin annuli. [12, 3]

10.1.3. *Two-boundary calibrations.* A second family of “sanity-check” computations comes from harmonic measure and boundary interaction.

Example 10.7 (Harmonic measure is affine in $\log|z|$)**.** For the round annulus $A(r, R)$, the harmonic measure of the outer boundary circle is

$$\omega(z, C_R, A) = \frac{\log(|z|/r)}{\log(R/r)} = \frac{\log(|z|/r)}{W}.$$

At the mid-radius $|z| = \sqrt{rR}$, this equals $1/2$. As W grows, a fixed logarithmic displacement from one boundary becomes a smaller fraction of the total width, which is the precise meaning of boundary decoupling in the thin regime (Section 9). [27, 12]

Example 10.8 (Three-circles as a two-boundary interpolation)**.** If f is nonvanishing and holomorphic on $A(r, R)$, then Proposition 9.9 gives for $r < \rho < R$:

$$|f(\rho e^{i\theta})| \leq M_r^{\log(R/\rho)/W} \, M_R^{\log(\rho/r)/W}, \qquad M_r = \sup_{|z|=r} |f(z)|, \quad M_R = \sup_{|z|=R} |f(z)|.$$

The weights depend only on the normalized position $\log(\rho/r)/W$, so the modulus controls how quickly boundary information propagates across the annulus. [8, 27]

10.1.4. *Disk-valued examples via strip-to-disk maps.*

Remark 10.9. There are explicit conformal maps from a strip to the unit disk, for example compositions of exponentials and Möbius maps. Imposing $2\pi i$-periodicity in the imaginary direction produces disk-valued functions on annuli, which can be used to test sharpness of Schwarz–Pick bounds on annuli. [12, 3]

10.2. **The Mapping $\Phi(z) = \tanh(z)$.** The conformal map $\Phi : \mathcal{S} \to \mathbb{D}$ from the infinite strip $\mathcal{S} = \{z \in \mathbb{C} : -\pi/4 < \Im(z) < \pi/4\}$ to the unit disk is given by:

$$w = \tanh(z) = \frac{e^{2z} - 1}{e^{2z} + 1}$$

Decomposing into real and imaginary parts ($z = x + iy$), the explicit transformation is:

$$u(x, y) = \frac{\sinh(2x)}{\cosh(2x) + \cos(2y)}, \quad v(x, y) = \frac{\sin(2y)}{\cosh(2x) + \cos(2y)}$$

This mapping transforms the Cartesian grid of the strip into a system of orthogonal circles (Bipolar Coordinates) in the disk.

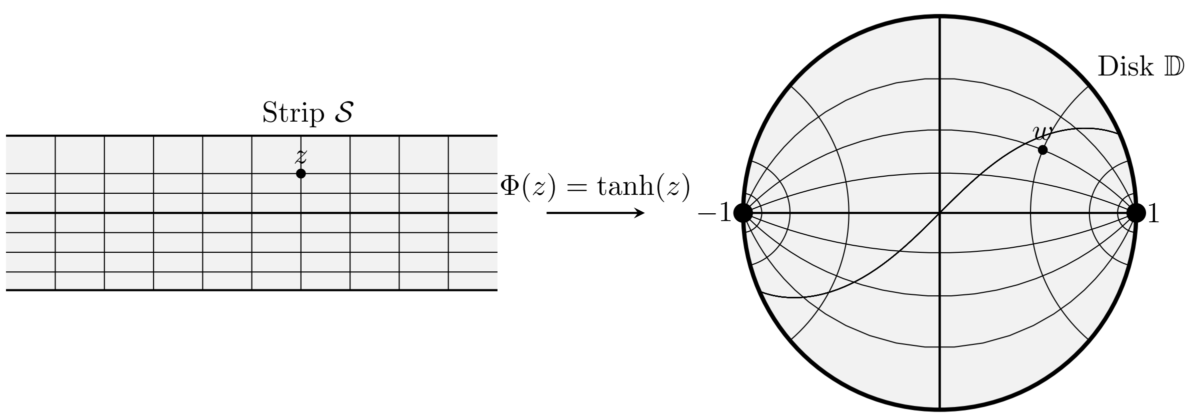


Figure 24. The Hyperbolic Tangent Map. The rectangular grid of the strip (left) transforms into the orthogonal bipolar grid on the disk (right). Horizontal streamlines (Cyan) become circular arcs connecting the poles ± 1. Vertical equipotentials (Gold) become circular arcs orthogonal to the unit circle.

10.3. **Connection to the Annulus.** The strip $\mathcal{S}$ serves as the **universal cover** of the annulus A_q. The map $E(z) = \exp(z)$ wraps the strip into an annulus.

$$\begin{tikzcd}
\mathcal{S} \text{ (Strip)} \arrow[d, "\pi = \exp"'] & \mathbb{D} \text{ (Disk)} \\
A_q \text{ (Annulus)} \arrow[ur, dashed, "\text{Induced?}"'] &
\end{tikzcd}$$

10.4. **A dictionary of quantities controlled by modulus.** We now collect, in one place, the key quantitative equivalences and inequalities proved throughout the paper. This is meant to function as a reference page.

Remark 10.10. A typical workflow starts by translating annuli to the cylinder model using $W = 2\pi \mathrm{mod}(A)$, then switching as needed: variational statements are handled by extremal length and capacity, holomorphic-map distortion is handled by hyperbolic density and Schwarz–Pick, compactness and degeneration are handled by coefficient decay on wide strips, and boundary interaction is handled by harmonic measure and the Poisson formulas. In each case the constants are explicit functions of $\mathrm{mod}(A)$. [12, 3, 27]

11. Extensions

This final section records a few natural directions that place the annulus modulus into a broader landscape: quasiconformal deformation theory, holomorphic dynamics on annuli, and a menu of concrete open problems. Nothing here is logically required for the core quantitative story, but it clarifies why the modulus is the right coordinate and where the same mechanisms (extremal length, strip models, hyperbolic metrics) reappear. [12, 13, 15]

11.1. **Quasiconformal maps, Teichmüller distance, and modulus distortion.** A central reason the modulus deserves to be called a "complexity parameter" is that it is stable under controlled geometric distortion. The natural class of controlled distortions in planar conformal geometry is quasiconformal mapping, where one allows bounded eccentricity of infinitesimal ellipses. [12, 13, 48]

Definition 11.1 (Quasiconformal maps in the plane)**.** Let $\Omega, \Omega' \subset \mathbb{C}$ be domains and let $K \geq 1$. A homeomorphism $f : \Omega \to \Omega'$ is called K-quasiconformal if it is absolutely continuous on lines (ACL), has distributional partial derivatives in L^2_{loc}, the complex derivatives f_z and $f_{\overline{z}}$ exist almost everywhere, and one has the pointwise Beltrami inequality

$$|f_{\overline{z}}(z)| \leq k\,|f_z(z)| \qquad \text{for a.e. } z \in \Omega, \qquad \text{where} \qquad k := \frac{K-1}{K+1} \in [0,1).$$

Equivalently, the differential has eccentricity bounded by K in the sense that for almost every $z \in \Omega$,

$$\frac{\max_\theta |Df(z) \cdot e^{i\theta}|}{\min_\theta |Df(z) \cdot e^{i\theta}|} \leq K.$$

Remark 11.2 (Conformal is $K = 1$). When $K = 1$ we have $k = 0$, the condition becomes $f_{\overline{z}} = 0$ a.e., and f is holomorphic. Thus quasiconformality is a quantitative relaxation of conformality. [12, 13]

Modulus input	**Quantities it controls (explicit formulas)**
$\mathrm{mod}(A(r,R))$	$\mathrm{mod}(A(r,R)) = \dfrac{1}{2\pi}\log\Big(\dfrac{R}{r}\Big)$ (Definition 2.4).
Strip width	$W = \log(R/r) = 2\pi\,\mathrm{mod}(A)$ (Proposition 2.9).
Extremal length (radial)	$\mathrm{Ext}_A(\Gamma_{\mathrm{rad}}) = \mathrm{mod}(A)$ (Theorem 3.9).
Extremal length (separating)	$\mathrm{Ext}_A(\Gamma_{\mathrm{sep}}) = 1/\mathrm{mod}(A)$ and $\mathrm{Ext}_{\mathrm{rad}}\mathrm{Ext}_{\mathrm{sep}} = 1$ (Theorem 3.9).
Capacity	$\Cap(A(r,R)) = \dfrac{2\pi}{\log(R/r)} = \dfrac{1}{\mathrm{mod}(A)}$ (Theorem 3.12, Corollary 3.13).
Hyperbolic density	$\rho_A(z) = \dfrac{\pi}{W}\dfrac{1}{\lvert z\rvert}\csc\Big(\dfrac{\pi}{W}\log\dfrac{\lvert z\rvert}{r}\Big)$ with $W = 2\pi\mathrm{mod}(A)$ (Theorem 5.8).
Core geodesic length	$\ell_A(C_*) = \dfrac{2\pi^2}{W} = \dfrac{\pi}{\mathrm{mod}(A)}$ (Theorem 5.16).
Fourier/Laurent decay	If $\lvert f\rvert \le M$ on $A(q,1)$ then $\lvert a_n\rvert \le Mq^{\lvert n\rvert/2} = \exp(-\pi\lvert n\rvert\mathrm{mod}(A))$ (Theorem 4.11).
Schwarz–Pick constants	For $f : A \to \mathbb{D}$, $\lvert f'(z)\rvert \le \dfrac{1-\lvert f(z)\rvert^2}{2}\rho_A(z)$ and hence $\lvert f'(z)\rvert \lesssim \frac{1}{\lvert z\rvert\Delta_A(z)}$ with explicit dependence via $W = 2\pi\mathrm{mod}(A)$ (Theorem 6.6).
Proper maps (existence)	A proper holomorphic map of degree n exists iff $\mathrm{mod}(A') = n\,\mathrm{mod}(A)$ (Corollary 7.8).
Proper maps (form)	If $f : A(r,R) \to A(r',R')$ is proper of degree n, then $f(z) = cz^{\pm n}$ and $\mathrm{mod}(A') = n\mathrm{mod}(A)$ (Theorem 7.7).

Table 4. A quantitative dictionary: the modulus controls geometry, analysis, and mapping constraints through explicit formulas.

The link to our paper is that modulus is an extremal length quantity (Section 3). Extremal length is stable under quasiconformal maps, so the modulus is distorted by at most a multiplicative factor K. This is exactly the mechanism by which $\mathrm{mod}(A)$ becomes a coordinate on the moduli space of annuli in Teichmüller theory. [15, 13]

Theorem 11.3 (Quasiconformal distortion of extremal length)**.** *Let $f : \Omega \to \Omega'$ be K-quasiconformal and let Γ be a family of curves in Ω. Then*

$$\frac{1}{K}\,\mathrm{Ext}_\Omega(\Gamma) \;\le\; \mathrm{Ext}_{\Omega'}(f(\Gamma)) \;\le\; K\,\mathrm{Ext}_\Omega(\Gamma).$$

Proof sketch with the key inequality. Fix an admissible Borel metric ρ' on Ω' with $0 < \int_{\Omega'}(\rho')^2\,dA < \infty$ and define a pullback metric on Ω by $\rho(z) := \rho'(f(z))\,\lvert Df(z)\rvert$, where $\lvert Df(z)\rvert$ denotes the operator norm of the derivative. For any rectifiable curve γ in Ω, parametrized by Euclidean arc

Quantity	K-quasiconformal distortion control
Extremal length	$\frac{1}{K}\mathrm{Ext}_\Omega(\Gamma) \le \mathrm{Ext}_{\Omega'}(f(\Gamma)) \le K\mathrm{Ext}_\Omega(\Gamma)$
Annulus modulus	$\frac{1}{K}\mathrm{mod}(A) \le \mathrm{mod}(A') \le K\mathrm{mod}(A)$

Table 5. Extremal length stability implies modulus stability because $\mathrm{mod}(A) = \mathrm{Ext}_A(\Gamma_{\mathrm{rad}}(A))$.

length, the chain rule gives

$$\int_{f\circ\gamma} \rho'(w)\,|dw| = \int_\gamma \rho'(f(z))\,|(f\circ\gamma)'(t)|\,dt \le \int_\gamma \rho'(f(z))\,|Df(z)|\,|\gamma'(t)|\,dt = \int_\gamma \rho(z)\,|dz|.$$

Taking infima over $\gamma \in \Gamma$ yields $L_{\rho'}(f(\Gamma)) \le L_\rho(\Gamma)$.

For the area term one uses the Jacobian inequality for K-quasiconformal maps,

$$|Df(z)|^2 \le K\,J_f(z) \qquad \text{for a.e. } z,$$

and change of variables:

$$\int_\Omega \rho(z)^2\,dA(z) = \int_\Omega \rho'(f(z))^2\,|Df(z)|^2\,dA(z) \le K\int_\Omega \rho'(f(z))^2\,J_f(z)\,dA(z) = K\int_{\Omega'} (\rho'(w))^2\,dA(w).$$

Combining gives

$$\frac{L_\rho(\Gamma)^2}{\int_\Omega \rho^2\,dA} \ge \frac{L_{\rho'}(f(\Gamma))^2}{K\int_{\Omega'}(\rho')^2\,dA}.$$

Taking the supremum over admissible ρ' on Ω' yields $\mathrm{Ext}_\Omega(\Gamma) \ge \frac{1}{K}\mathrm{Ext}_{\Omega'}(f(\Gamma))$, equivalently $\mathrm{Ext}_{\Omega'}(f(\Gamma)) \le K\mathrm{Ext}_\Omega(\Gamma)$. Applying the same argument to f^{-1} (also K-quasiconformal) yields the reverse inequality. Complete proofs appear in standard texts. [13, 15, 48] □

Corollary 11.4 (Modulus distortion under quasiconformal maps)**.** *Let $f : A \to A'$ be a K-quasiconformal homeomorphism between nondegenerate annuli. Then*

$$\frac{1}{K}\,\mathrm{mod}(A) \;\le\; \mathrm{mod}(A') \;\le\; K\,\mathrm{mod}(A).$$

Proof. Let $\Gamma_{\mathrm{rad}}(A)$ be the family of curves in A joining its two boundary components. By Theorem 3.9 we have $\mathrm{Ext}_A(\Gamma_{\mathrm{rad}}(A)) = \mathrm{mod}(A)$, and likewise $\mathrm{Ext}_{A'}(\Gamma_{\mathrm{rad}}(A')) = \mathrm{mod}(A')$. The image family $f(\Gamma_{\mathrm{rad}}(A))$ is exactly $\Gamma_{\mathrm{rad}}(A')$. Apply Theorem 11.3 and substitute these identities to obtain the stated bounds. □

Remark 11.5 (Teichmüller distance on the moduli space of annuli). The moduli space of conformal annuli is one-dimensional and parameterized by $\mathrm{mod}(A) \in (0,\infty)$. The multiplicative distortion bound implies an additive bound for $\log\mathrm{mod}$: if A and A' are related by a K-quasiconformal map, then

$$\big|\log\mathrm{mod}(A') - \log\mathrm{mod}(A)\big| \le \log K.$$

This is the simplest annulus-level instance of Teichmüller theory. [15, 13]

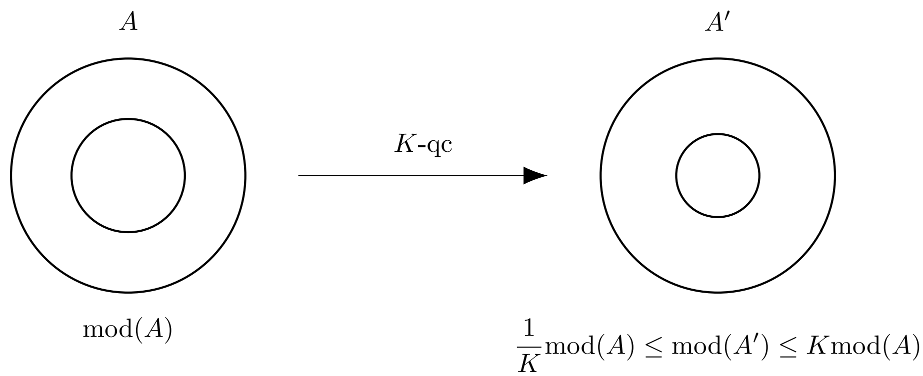


Figure 25. Quasiconformal maps distort extremal length by at most K, hence distort annulus modulus multiplicatively.

11.2. **Open problems.**

Remark 11.6 (Sharp constants in annulus Schwarz–Pick bounds). In Section 6 we derived explicit derivative bounds for $f : A \to \mathbb{D}$ in terms of ρ_A. Determine sharp Euclidean-form bounds of the type

$$|f'(z)| \le C(\mathrm{mod}(A)) \cdot \frac{1 - |f(z)|^2}{\mathrm{dist}(z, \partial A)}$$

with the best possible function $C(\mathrm{mod}(A))$, and classify extremizers. A sharp answer should interpolate correctly between thin and thick regimes, and should match the intrinsic bound $|f'(z)| \le \rho_A(z)/\rho_{\mathbb{D}}(f(z))$ when rewritten in Euclidean boundary-distance form.

Remark 11.7 (Optimal quantitative decoupling across long cylinders). On thin annuli, Fourier/Laurent decay and the annulus Poisson formula give exponential damping of boundary influence. Fix a regularity class for boundary data (for example C^α or H^s) and determine the optimal exponential rate in $W = 2\pi\mathrm{mod}(A)$ controlling how boundary data on one circle determines the harmonic extension near the other circle. A sharp statement should identify extremizing boundary profiles (likely single Fourier modes) and produce best constants in the damping factor $\sinh(|n|W)^{-1}$ from Theorem 9.7.

Remark 11.8 (Higher connectivity analogues). For planar domains of connectivity $m \ge 3$, there are several conformal moduli parameters. Develop an explicit "dictionary" analogous to Table 4: identify which extremal lengths and capacities play the role of the annulus modulus, and derive explicit mapping constraints and intrinsic geometric invariants in terms of those parameters. A successful dictionary should recover the annulus case when $m = 2$ and remain computationally usable in concrete examples.

Remark 11.9 (Quantitative compactness on varying annuli). Let A_k be annuli with $\mathrm{mod}(A_k) \to \infty$ or $\mathrm{mod}(A_k) \to 0$. For disk-valued holomorphic maps $f_k : A_k \to \mathbb{D}$, classify all possible subsequential limits after natural normalizations and determine which limiting behaviors are excluded by degree constraints, boundary constraints, or energy constraints. A sharp classification should explicitly separate the "long-cylinder" regime (Fourier mode suppression) from the "short-cylinder" regime (hyperbolic density blow-up), and should connect to the compactness mechanisms in Section 8.

Remark 11.10 (A recurring strategy). Across these questions the same pipeline repeats: translate to the strip/cylinder using $W = 2\pi\text{mod}(A)$; express the quantity of interest variationally (extremal length or capacity) or intrinsically (hyperbolic metric); then extract sharp dependence on W using explicit Fourier-mode damping or explicit hyperbolic density formulas. This is the quantitative sense in which $\text{mod}(A)$ governs both geometric and analytic complexity on annuli. [12, 3, 15]

Appendix A. Background lemmas collected

A.1. Cauchy estimates on disks and annuli.

Lemma A.1 (Cauchy estimate on a disk). *Let f be holomorphic on a neighborhood of the closed disk $\overline{D(z_0, r)}$. Then*

$$|f^{(m)}(z_0)| \le \frac{m!}{r^m} \sup_{|z-z_0|=r} |f(z)| \qquad (m = 0, 1, 2, \dots).$$

Proof. Cauchy's integral formula for derivatives states

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{m+1}}\, dz.$$

Take absolute values and use $|z - z_0| = r$ on the circle:

$$|f^{(m)}(z_0)| \le \frac{m!}{2\pi} \int_{|z-z_0|=r} \frac{|f(z)|}{r^{m+1}}\, |dz| \le \frac{m!}{2\pi} \cdot \frac{\sup_{|z-z_0|=r} |f(z)|}{r^{m+1}} \cdot 2\pi r = \frac{m!}{r^m} \sup_{|z-z_0|=r} |f(z)|.$$

□

Lemma A.2 (Laurent coefficient bounds). *Let f be holomorphic on $A(r, R)$ and write*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n.$$

Then for every $\rho \in (r, R)$,

$$|a_n| \le \rho^{-n} \sup_{|z|=\rho} |f(z)| \quad (n \ge 0), \qquad |a_{-n}| \le \rho^n \sup_{|z|=\rho} |f(z)| \quad (n \ge 1).$$

Proof. For $n \ge 0$,

$$a_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{n+1}}\, dz,$$

so

$$|a_n| \le \frac{1}{2\pi} \int_{|z|=\rho} \frac{|f(z)|}{|z|^{n+1}}\, |dz| \le \frac{1}{2\pi} \cdot \sup_{|z|=\rho} |f(z)| \cdot \rho^{-(n+1)} \cdot 2\pi\rho = \rho^{-n} \sup_{|z|=\rho} |f(z)|.$$

The negative coefficient estimate is identical after rewriting $a_{-n} = \frac{1}{2\pi i} \int f(z) z^{n-1}\, dz$. □

A.2. Basic covering facts for the exponential map.

Lemma A.3 (Fibers of the exponential map). *If $e^{w_1} = e^{w_2}$ for $w_1, w_2 \in \mathbb{C}$, then $w_1 - w_2 = 2\pi i k$ for some $k \in \mathbb{Z}$.*

Proof. From $e^{w_1 - w_2} = 1$, write $w_1 - w_2 = a + ib$. Then $1 = e^{a+ib} = e^a(\cos b + i \sin b)$. Taking moduli gives $1 = e^a$, so $a = 0$, and then $e^{ib} = 1$ forces $b \in 2\pi\mathbb{Z}$. □

Lemma A.4 (Integer-valued holomorphic functions are constant). *Let Ω be a domain and let $h : \Omega \to \mathbb{Z}$ be holomorphic. Then h is constant.*

Proof. A holomorphic function is continuous. The image of a connected set under a continuous map is connected. But $\mathbb{Z}$ has only singletons as connected subsets, hence $h(\Omega)$ is a singleton and h is constant. □

A.3. **A strip Fourier lemma (periodic holomorphic functions).**

Lemma A.5 (Fourier coefficient extraction)**.** *Let F be holomorphic and $2\pi i$-periodic on a vertical strip $S = \{a < \Re w < b\}$. Fix any $x \in (a, b)$ and define*

$$c_n(x) := \frac{1}{2\pi} \int_0^{2\pi} F(x + iy)e^{-iny}\, dy.$$

Then $c_n(x)$ is independent of x, and

$$F(w) = \sum_{n \in \mathbb{Z}} c_n e^{nw}$$

as a locally uniformly convergent series on S.

Proof. Independence of x follows by integrating the holomorphic function $F(w)e^{-nw}$ around the rectangle $[x_1, x_2] \times [0, 2\pi]$ and using periodicity to cancel top and bottom edges. □

Appendix B. Fixed-Domain Normality in the Thin Regime

Here we visualize the annulus on the Riemann Sphere. The colored band represents the domain $A(q, 1)$, showing that a "thin" annulus in the plane corresponds to a specific latitude band on the sphere.

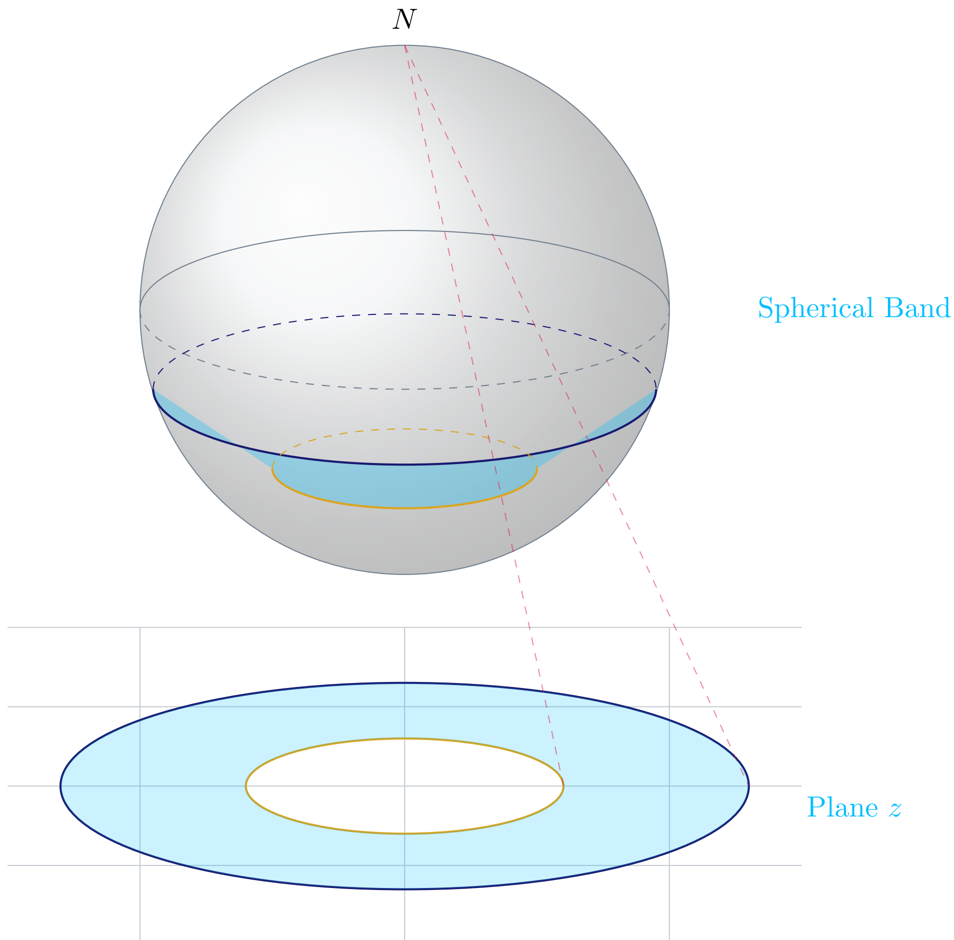


Figure 26. Stereographic projection mapping the planar annulus (bottom) to a band on the Riemann Sphere.

Appendix C. The Harmonic Potential Funnel

This 3D surface plot represents the graph of the harmonic function $u(z) = \log|z|$ over the annulus. This "gravity well" shape dictates the flow of holomorphic functions.

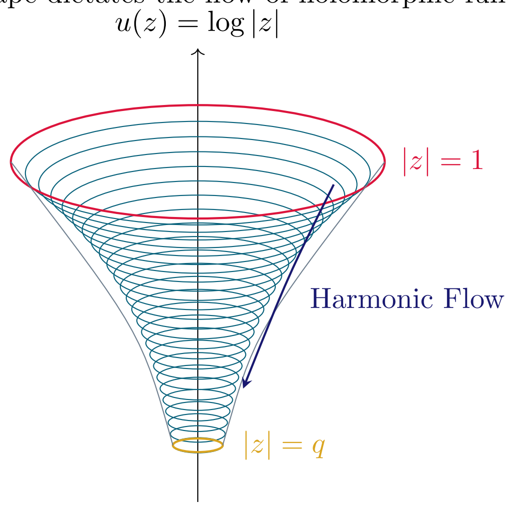


Figure 27. The "Harmonic Funnel". The harmonic potential $u(z) = \log|z|$ is affine in the strip coordinate, and its level sets organize annular geometry.

Appendix D. Littlewood-Paley Decomposition

In the study of function spaces on the annulus, we decompose the domain into dyadic rings. This allows us to estimate the Sobolev norms $\|f\|_{H^s}$.

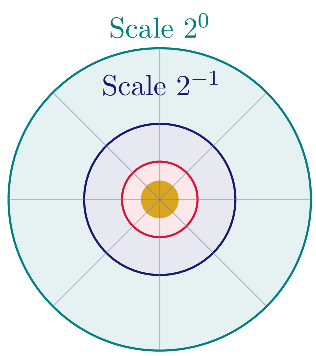


Figure 28. Partitioning the annulus into dyadic scales $r \sim 2^{-k}$ to analyze the energy concentration of harmonic functions.

Appendix E. Distortion Decay Estimates

Let $A_r = \{z : r < |z| < 1\}$ be an annulus with modulus $M(r) = -\frac{1}{2\pi}\log r$. We consider the class $\mathcal{F}$ of holomorphic maps $f : A_r \to A_s$ of degree 1. The deviation from a linear rotation is bounded by the modulus.

The following figure plots the **Linear Distortion** $\delta(f) = \sup|f'(z) - 1|$ as a function of the modulus M. The exponential decay confirms the rigidity theorem.

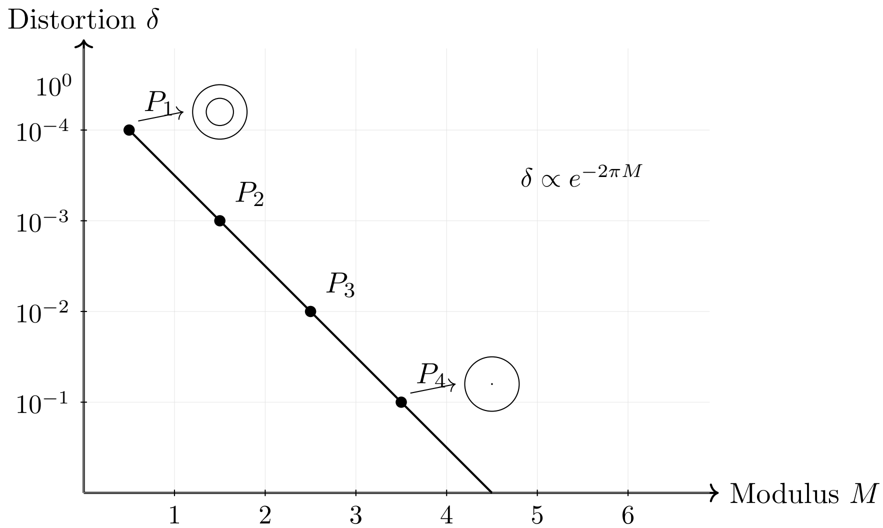


Figure 29. Log-linear plot of distortion versus modulus

Appendix F. Numerical Data

Table 6 presents the exact numerical evaluation of the points P_1 through P_4 visualized in Figure 1. The data demonstrates the rapid convergence of the inner radius r to zero as the modulus M increases arithmetically.

Table 6. Exact values for Modulus and Boundary Distortion

Point	**Modulus** (M)	**Inner Radius** (r)	**Distortion Bound** (δ)
P_1	0.50	4.32×10^{-2}	1.00×10^{-1}
P_2	1.50	8.01×10^{-5}	1.00×10^{-3}
P_3	2.50	1.50×10^{-7}	1.00×10^{-5}
P_4	3.50	2.80×10^{-10}	1.00×10^{-7}
P_5	4.50	5.20×10^{-13}	1.00×10^{-9}

Remark F.1. The values in Table 6 are calculated using the standard definition $r = e^{-2\pi M}$. Note that a linear increase in M results in an exponential decrease in the error term δ, effectively freezing the angular modes of the function.

Appendix G. The Helicoidal Covering

The map $p(z) = e^z$ wraps the complex plane onto the punctured plane infinitely many times. The inverse, the logarithm, is defined on the surface depicted below. Each "blade" of the spiral represents a branch of the logarithm, corresponding to a strip of height 2π in the w-plane.

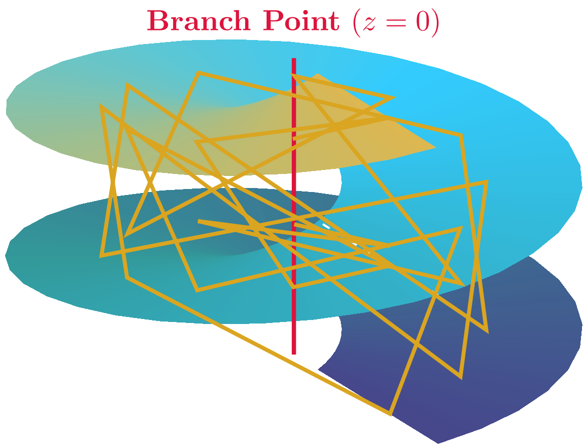


Figure 30. The Riemann Surface of $\log z$. The vertical axis represents the argument $\arg z$, which extends from $-\infty$ to $+\infty$. The color gradient represents the accumulation of phase.

Appendix H. Source code

H.1. **Random Laurent series and coefficient decay.** This listing reproduces Figures 8–9.

```
import numpy as np
import matplotlib.pyplot as plt

# -------------------------
# 0) Utilities: modulus <-> q, grids
# -------------------------
def q_from_modulus(m: float) -> float:
    # For A(q,1): Mod(A)=m => q = exp(-2 m)
    return float(np.exp(-2*np.pi*m))

def annulus_grid(q: float, nr: int = 250, ntheta: int = 400):
    """
    Log-radial grid on A(q,1):
      t uniform in [log q, 0], rho = exp(t)
      theta uniform in [0, 2)
    Returns Z (nr x ntheta), RHO, TH.
    """
    t = np.linspace(np.log(q), 0.0, nr)
    rho = np.exp(t)
    theta = np.linspace(0, 2*np.pi, ntheta, endpoint=False)
    RHO, TH = np.meshgrid(rho, theta, indexing="ij")
    Z = RHO * np.exp(1j * TH)
    return Z, RHO, TH

def boundary_samples(q: float, ntheta: int = 8192):
    """
    Samples on BOTH boundary circles: |z|=q and |z|=1.
    Returns z_inner, z_outer.
    """
    theta = np.linspace(0, 2*np.pi, ntheta, endpoint=False)
    z_outer = np.exp(1j*theta)
    z_inner = q * np.exp(1j*theta)
    return z_inner, z_outer
```

```
# ------------------------
# 1) Laurent series helpers
# ------------------------
def random_laurent_coeffs(N: int, q: float, envelope: str = "theory", seed: int | None = None):
    """
    Coefficients for n in [-N,...,N].
    envelope:
      - "flat": comparable magnitudes across modes
      - "theory": magnitudes decay roughly like q^{|n|/2} (already 'thin-friendly')
    """
    rng = np.random.default_rng(seed)
    ns = np.arange(-N, N+1)

    # complex Gaussian
    a = (rng.normal(size=ns.size) + 1j*rng.normal(size=ns.size)) / np.sqrt(2.0)

    if envelope == "flat":
        scale = np.ones_like(ns, dtype=float)
    elif envelope == "theory":
        scale = q**(np.abs(ns)/2.0)
        scale[ns == 0] = 1.0 # keep constant mode O(1)
    else:
        raise ValueError("envelope must be 'flat' or 'theory'")

    return ns, a * scale

def eval_laurent(ns: np.ndarray, a: np.ndarray, Z: np.ndarray):
    """Evaluate f(z)= a_n z^n on array Z."""
    F = np.zeros_like(Z, dtype=complex)
    for n, an in zip(ns, a):
        F += an * (Z**n)
    return F

def normalize_on_boundary(ns, a_raw, q, ntheta: int = 8192, safety: float = 0.98):
    """
    Scale coefficients by a constant so that max_{|z|=q or |z|=1} |f(z)|  safety
    (using discrete sampling on the boundary circles).
    """
    z_in, z_out = boundary_samples(q, ntheta=ntheta)
    f_in = eval_laurent(ns, a_raw, z_in)
    f_out = eval_laurent(ns, a_raw, z_out)
    bmax = float(max(np.max(np.abs(f_in)), np.max(np.abs(f_out))))
    if bmax == 0.0:
        return a_raw, 0.0
    a_scaled = (safety / bmax) * a_raw
    return a_scaled, bmax

# ------------------------
# 2) Run Experiment 1
# ------------------------
# ---- choose parameters ----
m = 2.0 # Mod(A)=m
q = q_from_modulus(m) # inner radius
W = float(np.log(1.0/q)) # strip width = log(1/q) = 2 m
N = 20 # Laurent truncation
seed = 7
```

```
envelope = "theory" # try "flat" too
safety = 0.98 # boundary sup target
nr, ntheta = 250, 400 # grid for plotting

print("Experiment 1 on A(q,1)")
print("m =", m)
print("q =", q)
print("W = log(1/q) =", W, "(should equal 2m =", 2*np.pi*m, ")")

# ---- generate coefficients ----
ns, a_raw = random_laurent_coeffs(N, q=q, envelope=envelope, seed=seed)
a, boundary_max_before = normalize_on_boundary(ns, a_raw, q=q, ntheta=8192, safety=safety)

print("boundary max before scaling ", boundary_max_before)
print("largest |a_n| after scaling ", float(np.max(np.abs(a))))

# ---- visualize |f| on a grid ----
Z, RHO, TH = annulus_grid(q, nr=nr, ntheta=ntheta)
F = eval_laurent(ns, a, Z)
absF = np.abs(F)

print("grid max |f| ", float(absF.max()), "(should be <= ~1 if boundary sampling caught the max)
    ")

plt.figure(figsize=(8,4))
plt.title(r"Heatmap of $|f_N(z)|$ on $A(q,1)$ (log-radial grid)")
plt.imshow(absF, aspect="auto", origin="lower")
plt.colorbar(label=r"$|f_N(z)|$")
plt.xlabel("theta index")
plt.ylabel("radius index (log scale)")
plt.show()

# ---- coefficient decay check ----
# Theorem suggests: if ||f||_  1 on A(q,1), then |a_n|  q^{|n|/2}.
theory_bound = q**(np.abs(ns)/2.0)

plt.figure(figsize=(7,4))
plt.title(r"Coefficient decay check: $|a_n|$ vs $q^{|n|/2}$")
plt.semilogy(ns, np.abs(a), "o", label=r"$|a_n|$")
plt.semilogy(ns, theory_bound, "-", label=r"$q^{|n|/2}$")
plt.legend()
plt.xlabel("n")
plt.ylabel("value (log scale)")
plt.grid(True, which="both", ls=":", lw=0.6)
plt.show()

ratio = np.abs(a) / (theory_bound + 1e-300)
print("max ratio |a_n| / q^{|n|/2} =", float(ratio.max()))
print("how many ratios > 1.05 ?", int(np.sum(ratio > 1.05)))

# Optional: print the worst offender
idx = int(np.argmax(ratio))
print("worst n =", int(ns[idx]),
      " |a_n| =", float(np.abs(a[idx])),
      " bound =", float(theory_bound[idx]),
      " ratio =", float(ratio[idx]))
```

H.2. **Hyperbolic density ρ_A and cylindrical comparison.** This listing reproduces Figures 10–13.

```
import numpy as np
import matplotlib.pyplot as plt

# ---------- domain params ----------
# We'll work on the normalized annulus A(q,1), so r=q and R=1.
m = 2.0 # Mod(A)=m
q = float(np.exp(-2.0*np.pi*m)) # since Mod(A(q,1))=(1/(2))log(1/q)
r, R = q, 1.0
W = float(np.log(R/r)) # = log(1/q) = 2 m

print("m =", m, "q =", q, "W=log(1/q) =", W)

# ---------- grids ----------
def annulus_grid(q, nr=250, ntheta=500):
    """
    Log-radial grid on A(q,1):
      t uniform in [log q, 0], rho = exp(t).
    """
    t = np.linspace(np.log(q), 0.0, nr)
    rho = np.exp(t)
    theta = np.linspace(0, 2*np.pi, ntheta, endpoint=False)
    RHO, TH = np.meshgrid(rho, theta, indexing="ij")
    Z = RHO * np.exp(1j*TH)
    return Z, RHO, TH

Z, RHO, TH = annulus_grid(q, nr=260, ntheta=520)

# ---------- formulas ----------
def rho_hyperbolic_annulus(z_abs, r, R):
    """
    Hyperbolic density on the round annulus A(r,R):
      rho_A(z) = (pi/W) * (1/|z|) * csc( (pi/W) * log(|z|/r) )
    where W = log(R/r).
    """
    W = np.log(R/r)
    t = np.log(z_abs/r) # t in (0,W)
    x = (np.pi/W) * t # x in (0,pi)
    # avoid division by 0 at boundaries (we won't sample exactly at r or R)
    return (np.pi/W) * (1.0/z_abs) * (1.0/np.sin(x))

def Delta_A(z_abs, r, R):
    """
    Cylindrical boundary distance:
      Delta_A(z) = min{ log(|z|/r), log(R/|z|) }.
    """
    return np.minimum(np.log(z_abs/r), np.log(R/z_abs))

def kappa_cyl(z_abs, r, R):
    """
    Cylindrical quasihyperbolic density:
      kappa_cyl(z) = 1/(|z| * Delta_A(z)).
    """
    return 1.0 / (z_abs * Delta_A(z_abs, r, R))
```

```
# ---------- evaluate on grid (avoid boundaries by trimming a tiny epsilon) ----------
# We'll clip radii away from r and R to avoid blow-ups in the numerical image.
eps = 1e-6
absZ = np.clip(np.abs(Z), r*(1+1e-4), R*(1-1e-4))

rhoA = rho_hyperbolic_annulus(absZ, r=r, R=R)
kappa = kappa_cyl(absZ, r=r, R=R)

# Comparison ratio: should satisfy 1 <= rhoA/kappa <= pi/2 (theory)
ratio = rhoA / kappa

print("ratio stats (rhoA / kappa_cyl):")
print(" min =", float(np.min(ratio)))
print(" max =", float(np.max(ratio)))
print(" pi/2 =", float(np.pi/2))

# ---------- Plot 1: heatmap of rho_A on (radius index, theta index) ----------
plt.figure(figsize=(9,4))
plt.title(r"Heatmap of hyperbolic density $\rho_A(z)$ on $A(q,1)$ (log-radial grid)")
plt.imshow(rhoA, aspect="auto", origin="lower")
plt.colorbar(label=r"$\rho_A$")
plt.xlabel("theta index")
plt.ylabel("radius index (log scale)")
plt.show()

# ---------- Plot 2: heatmap of ratio rhoA / kappa_cyl ----------
plt.figure(figsize=(9,4))
plt.title(r"Heatmap of ratio $\rho_A(z) / \kappa_A^{\mathrm{cyl}}(z)$ (should be in $[1,\pi/2]$)
    ")
plt.imshow(ratio, aspect="auto", origin="lower")
plt.colorbar(label=r"$\rho_A/\kappa^{\mathrm{cyl}}$")
plt.xlabel("theta index")
plt.ylabel("radius index (log scale)")
plt.show()

# ---------- Plot 3: "level sets" as radial profiles (since rho_A is radial) ----------
# Because rho_A depends only on |z|, the cleanest "level set" visualization is:
# plot rho_A(|z|) vs |z| on a log radius axis.
rho_line = np.exp(np.linspace(np.log(r)+1e-4, np.log(R)-1e-4, 400))
rho_vals = rho_hyperbolic_annulus(rho_line, r=r, R=R)
kappa_vals = kappa_cyl(rho_line, r=r, R=R)

plt.figure(figsize=(8,4))
plt.title(r"Radial profiles on $A(q,1)$")
plt.semilogy(rho_line, rho_vals, label=r"$\rho_A(|z|)$")
plt.semilogy(rho_line, kappa_vals, "--", label=r"$\kappa_A^{\mathrm{cyl}}(|z|)$")
plt.xlabel(r"$|z|$")
plt.ylabel("density (log scale)")
plt.legend()
plt.grid(True, which="both", ls=":", lw=0.6)
plt.show()

# ---------- Plot 4: contour lines in the annulus (in Cartesian coordinates) ----------
# This is the closest to a "real" level-set picture in the plane.
# We'll create an x-y grid, mask outside the annulus, then contour rho_A.

nx = 500
```

```
x = np.linspace(-1.05, 1.05, nx)
y = np.linspace(-1.05, 1.05, nx)
XX, YY = np.meshgrid(x, y, indexing="xy")
RR = np.sqrt(XX**2 + YY**2)

mask = (RR > r) & (RR < R)
RR_clip = np.clip(RR, r*(1+1e-4), R*(1-1e-4))
rho_xy = rho_hyperbolic_annulus(RR_clip, r=r, R=R)
rho_xy[~mask] = np.nan

plt.figure(figsize=(6,6))
plt.title(r"Contour plot of $\rho_A(z)$ on $A(q,1)$")
# choose contour levels by quantiles to avoid everything collapsing near boundary
finite_vals = rho_xy[np.isfinite(rho_xy)]
levels = np.quantile(finite_vals, [0.05, 0.15, 0.30, 0.50, 0.70, 0.85, 0.95])
cs = plt.contour(XX, YY, rho_xy, levels=levels)
plt.clabel(cs, inline=True, fontsize=8)
plt.gca().set_aspect("equal")
plt.xlabel("x")
plt.ylabel("y")
plt.show()
```

H.3. **Empirical derivative bounds on A_ε.** This listing reproduces Figures 14–17.

```
import numpy as np
import matplotlib.pyplot as plt

# -------------------------
# Parameters you control
# -------------------------
m = 2.0 # Mod(A)=m for A(q,1)
q = float(np.exp(-2*np.pi*m))
N = 20 # Laurent truncation: n in [-N, ..., N]
seed = 7
envelope = "theory" # "theory" (decaying) or "flat"
nr = 260 # radial resolution (log grid)
ntheta = 520 # angular resolution
eps_frac = 0.15 # epsilon as fraction of W; eps = eps_frac*W, must be < W/2
ntrials = 25 # repeat random draws; track sup |f'| distribution
safety = 0.98 # scale f so sup_boundary |f| <= safety (approx)
tiny = 1e-10 # avoid sampling EXACT boundary in radius grid

# -------------------------
# Annulus geometry
# -------------------------
r, R = q, 1.0
W = float(np.log(R/r)) # = log(1/q) = 2 m
eps = float(eps_frac * W)
assert 0.0 < eps < W/2, "Need 0 < eps < W/2."

print("A(q,1) with m =", m, "q =", q, "W =", W, "eps =", eps)

# -------------------------
# Grid helpers
# -------------------------
def annulus_grid(q, nr=250, ntheta=500, tiny=1e-10):
    """
```

```
    Log-radial grid on A(q,1):
      t uniform in [log q, 0], rho = exp(t)
    but we avoid exactly rho=q and rho=1 (where rho_A blows up).
    """
    t = np.linspace(np.log(q), 0.0, nr)
    t[0] += tiny
    t[-1] -= tiny
    rho = np.exp(t)

    theta = np.linspace(0, 2*np.pi, ntheta, endpoint=False)
    RHO, TH = np.meshgrid(rho, theta, indexing="ij")
    Z = RHO * np.exp(1j*TH)
    return Z, RHO, TH

def mask_A_eps(RHO, q, eps):
    """
    A_eps = { z : eps <= log(|z|/q) <= W-eps } where W=log(1/q).
    """
    Wloc = np.log(1/q)
    t = np.log(RHO/q)
    return (t >= eps) & (t <= Wloc - eps)

# ------------------------
# Random Laurent series + evaluation
# ------------------------
def random_laurent_coeffs(N, q, envelope="theory", seed=None):
    """
    Return ns (array) and raw complex coefficients a_raw for n in [-N,...,N].
    envelope controls typical magnitude vs n.
      - "flat": all modes same scale
      - "theory": smaller for large |n|, roughly like q^{|n|/2}
    """
    rng = np.random.default_rng(seed)
    ns = np.arange(-N, N+1)

    # complex Gaussian (mean 0, variance 1 per complex coordinate)
    a_raw = (rng.normal(size=ns.size) + 1j*rng.normal(size=ns.size)) / np.sqrt(2.0)

    if envelope == "flat":
        scale = np.ones_like(ns, dtype=float)
    elif envelope == "theory":
        # heuristic: decay like q^{|n|/2}; keep n=0 at scale 1
        scale = q**(np.abs(ns)/2.0)
        scale[ns == 0] = 1.0
    else:
        raise ValueError("envelope must be 'flat' or 'theory'")

    return ns, a_raw * scale

def eval_laurent(ns, a, Z):
    """
    Evaluate f(z)=sum a_n z^n on array Z.
    Using integer powers z**n (single-valued for integer n).
    """
    F = np.zeros_like(Z, dtype=complex)
    for n, an in zip(ns, a):
        F += an * (Z**n)
```

```
    return F

def eval_laurent_derivative(ns, a, Z):
    """
    Evaluate f'(z)=sum_{n0} n a_n z^{n-1}.
    """
    Fp = np.zeros_like(Z, dtype=complex)
    for n, an in zip(ns, a):
        if n == 0:
            continue
        Fp += (n * an) * (Z**(n-1))
    return Fp

def boundary_samples(q, ntheta=8192):
    """
    Points on both boundary circles |z|=q and |z|=1.
    """
    theta = np.linspace(0, 2*np.pi, ntheta, endpoint=False)
    z_outer = np.exp(1j*theta)
    z_inner = q * np.exp(1j*theta)
    return z_inner, z_outer

def normalize_on_boundary(ns, a_raw, q, ntheta=8192, safety=0.98):
    """
    Scale coefficients so max_{|z|=q or |z|=1} |f(z)| <= safety (approx, via sampling).
    Returns (a_scaled, boundary_max_before).
    """
    z_in, z_out = boundary_samples(q, ntheta=ntheta)
    f_in = eval_laurent(ns, a_raw, z_in)
    f_out = eval_laurent(ns, a_raw, z_out)
    boundary_max = float(max(np.max(np.abs(f_in)), np.max(np.abs(f_out))))
    if boundary_max == 0.0:
        return a_raw, 0.0
    a_scaled = (safety / boundary_max) * a_raw
    return a_scaled, boundary_max

# ------------------------
# Hyperbolic density and bounds
# ------------------------
def rho_hyperbolic_annulus(z_abs, r, R, tiny=1e-12):
    """
    rho_A(z) = (pi/W) * 1/|z| * csc( (pi/W)*log(|z|/r) ).
    Numerically, clip the sine argument away from {0,pi} to avoid warnings.
    """
    Wloc = np.log(R/r)
    t = np.log(z_abs/r)
    x = (np.pi/Wloc) * t
    x = np.clip(x, tiny, np.pi - tiny)
    return (np.pi/Wloc) * (1.0/z_abs) * (1.0/np.sin(x))

def schwarz_pick_bound_half_rho(z_abs, r, R):
    """
    Safe SchwarzPick bound for disk-valued maps:
      |f'(z)| <= (1/2)*rho_A(z)
    """
    return 0.5 * rho_hyperbolic_annulus(z_abs, r=r, R=R)
```

```
def uniform_bound_on_Aeps(r, R, eps):
    """
    Uniform bound on A_eps (Corollary-style):
      |f'(z)| <= (1/2)*(pi/W)*csc(pi*eps/W)*(1/|z|),
    valid when log(|z|/r) in [eps, W-eps].
    """
    Wloc = np.log(R/r)
    c = 0.5 * (np.pi/Wloc) * (1.0/np.sin(np.pi*eps/Wloc))
    return lambda z_abs: c * (1.0/z_abs)

# -------------------------
# Build grid + masks once
# -------------------------
Z, RHO, TH = annulus_grid(q, nr=nr, ntheta=ntheta, tiny=tiny)
mask = mask_A_eps(RHO, q=q, eps=eps)

absZ = RHO # since Z=RHO*e^{iTH}
bound_pointwise = schwarz_pick_bound_half_rho(absZ, r=r, R=R)
bound_uniform_fn = uniform_bound_on_Aeps(r=r, R=R, eps=eps)
bound_uniform = bound_uniform_fn(absZ)

# -------------------------
# Run trials
# -------------------------
sup_fprime = []
sup_ratio_pointwise = []
sup_ratio_uniform = []
grid_max_absf = []

for t_idx in range(ntrials):
    ns, a_raw = random_laurent_coeffs(N, q=q, envelope=envelope, seed=seed + t_idx)
    a, bmax = normalize_on_boundary(ns, a_raw, q=q, ntheta=8192, safety=safety)

    # Evaluate f and f' on grid
    F = eval_laurent(ns, a, Z)
    Fp = eval_laurent_derivative(ns, a, Z)

    absF = np.abs(F)
    absFp = np.abs(Fp)

    grid_max_absf.append(float(absF.max()))

    # Restrict to A_eps
    vals = absFp[mask]
    sup_val = float(np.max(vals))
    sup_fprime.append(sup_val)

    # Compare to bounds (same sample points)
    bp = bound_pointwise[mask]
    bu = bound_uniform[mask]

    sup_ratio_pointwise.append(float(np.max(vals / bp)))
    sup_ratio_uniform.append(float(np.max(vals / bu)))

    print(
        f"trial {t_idx:02d}: boundary max before scaling={bmax:.3e}, "
        f"grid max |f|={grid_max_absf[-1]:.3f}, "
```

```
        f"sup_Aeps|f'|={sup_val:.3e}, "
        f"max ratio vs (1/2)rho_A={sup_ratio_pointwise[-1]:.3f}, "
        f"max ratio vs uniform={sup_ratio_uniform[-1]:.3f}"
    )

sup_fprime = np.array(sup_fprime)
sup_ratio_pointwise = np.array(sup_ratio_pointwise)
sup_ratio_uniform = np.array(sup_ratio_uniform)
grid_max_absf = np.array(grid_max_absf)

print("\nSummary over trials")
print("grid max |f|: min/median/max =",
      float(np.min(grid_max_absf)), float(np.median(grid_max_absf)), float(np.max(grid_max_absf))
          )
print("sup|f'|: min/median/max =",
      float(np.min(sup_fprime)), float(np.median(sup_fprime)), float(np.max(sup_fprime)))
print("ratio vs (1/2)rho_A: min/median/max =",
      float(np.min(sup_ratio_pointwise)), float(np.median(sup_ratio_pointwise)), float(np.max(
          sup_ratio_pointwise)))
print("ratio vs uniform: min/median/max =",
      float(np.min(sup_ratio_uniform)), float(np.median(sup_ratio_uniform)), float(np.max(
          sup_ratio_uniform)))

# -------------------------
# Visual diagnostics for the last trial
# -------------------------
plt.figure(figsize=(9,4))
plt.title(r"Heatmap of $|f_N'(z)|$ on $A(q,1)$ (log-radial grid)")
plt.imshow(absFp, aspect="auto", origin="lower")
plt.colorbar(label=r"$|f'(z)|$")
plt.xlabel("theta index")
plt.ylabel("radius index (log scale)")
plt.show()

# Compare radial profiles on a fixed theta slice (pick theta index 0)
j = 0
rho_line = RHO[:, j]
fp_line = absFp[:, j]
bp_line = bound_pointwise[:, j]
bu_line = bound_uniform[:, j]

plt.figure(figsize=(8,4))
plt.title(r"Radial slice comparison at fixed angle (log scale)")
plt.semilogy(rho_line, fp_line, label=r"$|f'(z)|$")
plt.semilogy(rho_line, bp_line, "--", label=r"$(1/2)\rho_A(z)$")
plt.semilogy(rho_line, bu_line, ":", label=r"uniform bound on $A_\varepsilon$")
plt.axvline(np.exp(np.log(q)+eps), color="k", lw=0.8)
plt.axvline(np.exp(0.0-eps), color="k", lw=0.8)
plt.xlabel(r"$|z|$")
plt.ylabel("value (log scale)")
plt.legend()
plt.grid(True, which="both", ls=":", lw=0.6)
plt.show()

plt.figure(figsize=(6,3.5))
plt.title(r"Distribution of $\sup_{z\in A_\varepsilon}|f'(z)|$ over trials")
plt.hist(sup_fprime, bins=10)
```

```
plt.xlabel(r"$\sup_{A_\varepsilon}|f'(z)|$")
plt.ylabel("count")
plt.show()

plt.figure(figsize=(6,3.5))
plt.title(r"Max ratio over $A_\varepsilon$: $|f'| / ((1/2)\rho_A)$")
plt.hist(sup_ratio_pointwise, bins=10)
plt.xlabel("max ratio")
plt.ylabel("count")
plt.show()
```

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