CONWAY MUTATION AND THE JONES POLYNOMIAL

IZ TECÚM

Abstract. We study the behavior of the Kauffman bracket and the Jones polynomial under Conway mutation. Using the bracket vector formalism for 2–tangles, we give a self-contained proof that the Jones polynomial is invariant under mutation. We further discuss consequences for the Jones–unknot problem and explain why mutation cannot produce counterexamples. Classical examples such as the Conway and Kinoshita–Terasaka knots are used to illustrate the theory.

1. Introduction

Since its introduction by Jones in the mid–1980s [1], the Jones polynomial has played a central role in knot theory, providing one of the first powerful and computable invariants capable of distinguishing large classes of knots. Its discovery led rapidly to new connections between low–dimensional topology, statistical mechanics, and operator algebras, and inspired a wide range of subsequent knot invariants.

One of the earliest and most striking phenomena observed for the Jones polynomial is its invariance under Conway mutation [2, 4]. Conway mutation is a geometric operation that alters a knot in a highly controlled way: one cuts the knot along a Conway sphere—a 2–sphere intersecting the knot in four points—and reglues after a π–rotation of the enclosed 2–tangle. The resulting knot K' is called a *mutant* of K. Although K and K' may be topologically distinct, many classical invariants fail to distinguish them.

A particularly striking example is the Jones polynomial $V_K(t)$, obtained from the Kauffman bracket by the normalization

$$V_K(t) = (-A^3)^{-w(D)} \langle D \rangle \big|_{A=t^{-1/4}} .$$

Since both the Kauffman bracket and the writhe are preserved, the normalized bracket is unchanged, giving

$$V_K(t) = V_{K'}(t).$$

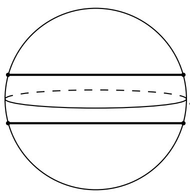
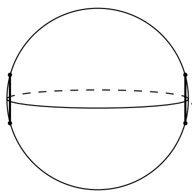
Thus the Jones polynomial cannot detect mutation, as exemplified by the Conway and Kinoshita–Terasaka knots.

This paper develops these ideas using explicit bracket-vector calculations and diagrammatic arguments showing how states change under rotation.

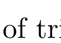
2. Definitions and Preliminaries

Notation 2.1. We fix two standard 2–tangles, denoted T_0 and T_∞. They are defined by

We fix the two trivial 2–tangles

$T_0 :=$ , $T_\infty :=$ ,

or

$T_0 :=$ , $T_\infty :=$ .

This choice follows the standard picture of trivial 2–tangles in the literature (see, for example, Kauffman [5, 2] and Lickorish [3]).

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Here, we write $\langle T_0 \rangle$ and $\langle T_\infty \rangle$ for their Kauffman brackets.

2.1. **The Kauffman bracket and Jones polynomial.**

Definition 2.1. Let D be an unoriented link diagram in the plane. The *Kauffman bracket* $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ is defined by [2, 5, 3]

(1) $\langle \varnothing \rangle = 1$.
(2) For a disjoint union with an unknotted circle,

$$\langle D \sqcup \bigcirc \rangle = (-A^2 - A^{-2}) \langle D \rangle,$$

and at a crossing

$$\left\langle \times \right\rangle = A \left\langle \asymp \right\rangle + A^{-1} \left\langle)(\right\rangle$$

$$\left\langle \times \right\rangle = A \left\langle)(\right\rangle + A^{-1} \left\langle \asymp \right\rangle.$$

(3) We will write D_0 for the diagram obtained from D by replacing that crossing with the *horizontal* (A–) smoothing $\asymp$, and D_∞ for the diagram obtained by replacing it with the *vertical* (A^{-1}–) smoothing $)($. Thus

$$\langle D \rangle = A \langle D_0 \rangle + A^{-1} \langle D_\infty \rangle.$$

Remark 2.1. Later, the notation D_0, D_∞ above will correspond to the notation for T_0, T_∞, implying that *whole link diagrams* are obtained from D by smoothing a single crossing into a T_0 or T_∞. This viewpoint is standard in the state–sum approach to the Jones polynomial [5, 3].

Definition 2.2. Let D be an oriented diagram of a knot K. The *writhe* $w(D) \in \mathbb{Z}$ is defined by

$$w(D) = \sum_c \operatorname{sgn}(c),$$

where the sum runs over all crossings c of D, and $\operatorname{sgn}(c) = +1$ at a positive crossing and -1 at a negative crossing (cf. [3, 6]).

Definition 2.3. Let D be an oriented diagram of a knot K. Define

$$f_D(A) = (-A^3)^{-w(D)} \langle D \rangle.$$

Then $f_D(A)$ is invariant under the three Reidemeister moves; hence it depends only on K and not on the chosen diagram. The *Jones polynomial* of K is

$$V_K(t) = f_D\big(t^{-1/4}\big) = (-A^3)^{-w(D)} \langle D \rangle \Big|_{A=t^{-1/4}} \in \mathbb{Z}[t^{\pm 1/2}].$$

This normalization is the original one introduced by Jones in [1] and is treated in detail in standard texts such as [3, 7].

Theorem 2.1. Let K be a knot and D, D' any two oriented diagrams (differing only by Reidemeister moves) of K. Then

$$(-A^3)^{-w(D)} \langle D \rangle = (-A^3)^{-w(D')} \langle D' \rangle.$$

In particular, $V_K(t)$ is well-defined as an invariant of K (see, for example, [1, 3, 7]).

2.2. **Conway spheres and Conway mutation.**

Definition 2.4. Let $K \subset S^3$ be a knot. An embedded 2–sphere $S \subset S^3$ is a *Conway sphere* for K if it meets K transversely in exactly four points. The intersections of K with the two 3–balls bounded by S are 2–tangles such that they partition K into two 2–tangles. Conway spheres and their role in mutation are treated, for example, in Kawauchi [8] and Morton [4].

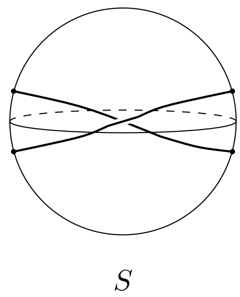


FIGURE 1. The Conway sphere S meeting a knot in four points.

Definition 2.5 (Conway mutation)**.** Let K be a knot and let S be a Conway sphere for K, so that S meets K in four points and cuts K into two 2–tangles,

$$K = F \cup G,$$

where F lies in one 3–ball bounded by S and G lies in the other.

There are three standard π–rotations of the ball containing F, each preserving the four boundary points of F; we denote them by ρ_1, ρ_2, ρ_3 (see Figure 2). For $i = 1, 2, 3$ let

$$F_i = \rho_i(F)$$

be the rotated 2–tangle with the same boundary as F. This is the usual definition of Conway mutation as found, for example, in Morton [4] and Cromwell [7].

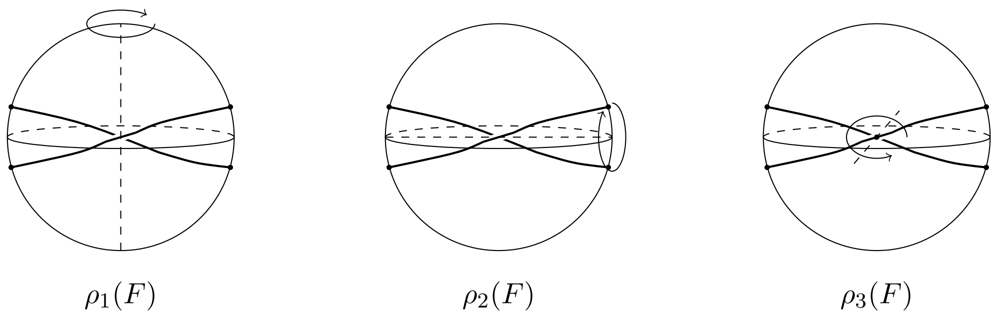


FIGURE 2. The three π–rotations $\rho_1(F), \rho_2(F), \rho_3(F)$ of a Conway sphere.

The knot

$$K' = F_i \cup G$$

obtained by replacing F with F_i is called a *mutant* of K. The operation $K \mapsto K'$ is called *Conway mutation*; see, for instance, Morton [4] and Cromwell [7] for further discussion and examples.

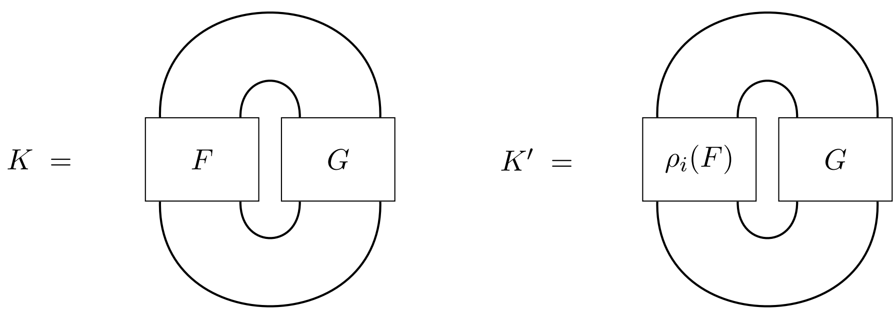


FIGURE 3. Forming the mutant $K' = \rho_i(F) \cup G$ by replacing F with $\rho_i(F)$ (cf. [4, 7]).

Example 2.1. The Conway knot C and the Kinoshita–Terasaka knot KT form a mutant pair. Each knot admits a Conway sphere whose π–rotation replaces one 2–tangle by its rotated copy such that

$$C \longleftrightarrow KT.$$

Interestingly, they have the same Jones polynomials,

$$V_C(t) = V_{KT}(t) = t^{-4}\big(-1 + 2t - 2t^2 + 2t^3 + t^6 - 2t^7 + 2t^8 - 2t^9 + t^{10}\big)\,,$$

a computation that can be found, for instance, in Morton [4] and is also discussed in expository texts such as Adams [6].

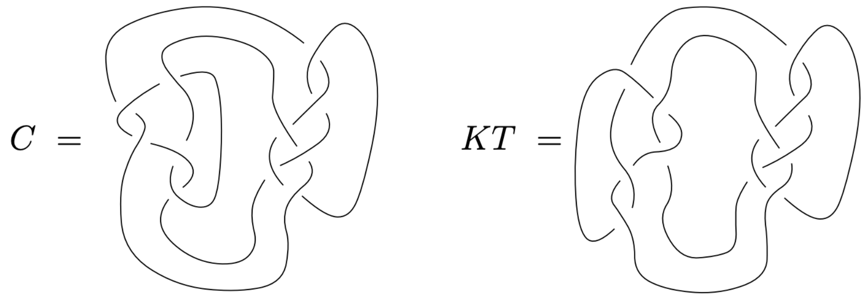


FIGURE 4. The Conway and Kinoshita–Terasaka knots, as shown in Morton [4], are a classic example of inequivalent mutant knots, sharing the same Jones polynomial.

This example demonstrates that the Jones polynomial does not detect mutation [4, 7].

3. Bracket Vectors for 2–Tangles

3.1. States and Weights.

Definition 3.1. Let T be a 2–tangle diagram. A *state* s of T is obtained by choosing at each crossing either the A–smoothing or the A^{-1}–smoothing, in the sense of Kauffman's state model for the bracket [2, 5].

For any state s we write

$$a(s) = \#\{A\text{–smoothings in } s\}, \qquad b(s) = \#\{A^{-1}\text{–smoothings in } s\}, \qquad \ell(s) = \#\{\text{loops in } s\}.$$

The contribution of s to the Kauffman bracket is

$$\text{weight}(s) = A^{\,a(s)-b(s)}\big(-A^2 - A^{-2}\big)^{\ell(s)}.$$

We denote by

$$s_0(T) = \{s \mid s \text{ has boundary connectivity } T_0\}, \qquad s_\infty(T) = \{s \mid s \text{ has boundary connectivity } T_\infty\}.$$

Define Laurent polynomials

$$f_T(A) = \sum_{s\in s_0(T)} \text{weight}(s), \qquad g_T(A) = \sum_{s\in s_\infty(T)} \text{weight}(s).$$

Then the Kauffman bracket of T is

$$\langle T\rangle = f_T(A)\,\langle T_0\rangle + g_T(A)\,\langle T_\infty\rangle,$$

and the pair

$$\text{br}(T) = \big(f_T(A), g_T(A)\big)$$

is called the *bracket vector* of the 2–tangle T (cf. Lickorish [3, §5] and Cromwell [7, Ch. 8]).

Example 3.1. Consider the 2–tangle T consisting of a single positive crossing:

$$\langle T \rangle \;=\; \left\langle \times \right\rangle.$$

By the Kauffman bracket skein relation we also have

$$\left\langle \times \right\rangle \;=\; A \left\langle \asymp \right\rangle + A^{-1} \left\langle)\,(\right\rangle.$$

There are exactly two states of T. In the first state, we choose the A–smoothing at the crossing. The resulting diagram has boundary arcs of type T_0 (two horizontal strands, as in $T_0 = \asymp$), so this state lies in $s_0(T)$. Here

$$a(s) = 1, \qquad b(s) = 0, \qquad \ell(s) = 0,$$

and therefore

$$\text{weight}(s) = A^{a(s)-b(s)} \big(-A^2 - A^{-2}\big)^{\ell(s)} = A.$$

In the second state, we choose the A^{-1}–smoothing at the crossing. The resulting diagram has boundary arcs of type T_∞ (two vertical strands, as in $T_\infty = \big|\;\big|$), so this state lies in $s_\infty(T)$. In this case

$$a(s) = 0, \qquad b(s) = 1, \qquad \ell(s) = 0,$$

and hence

$$\text{weight}(s) = A^{a(s)-b(s)} \big(-A^2 - A^{-2}\big)^{\ell(s)} = A^{-1}.$$

Since each of the sets $s_0(T)$ and $s_\infty(T)$ contains exactly one state, we obtain

$$f_T(A) = \sum_{s \in s_0(T)} \text{weight}(s) = A, \qquad g_T(A) = \sum_{s \in s_\infty(T)} \text{weight}(s) = A^{-1},$$

so the bracket vector is

$$\text{br}(T) = \big(f_T(A), g_T(A)\big) = (A, A^{-1}),$$

and the bracket decomposes as

$$\left\langle \times \right\rangle = A\,\langle T_0 \rangle + A^{-1}\,\langle T_\infty \rangle.$$

This matches the skein relation and illustrates how the bracket vector records the contributions of the two boundary connectivity types.

3.2. **Decomposition of the Bracket.**

Lemma 3.1 (Bracket vector decomposition)**.** Let T be a 2–tangle diagram, and let $f_T(A), g_T(A)$ be the Laurent polynomials defined by

$$f_T(A) := \sum_{s \in s_0(T)} \text{weight}(s), \qquad g_T(A) := \sum_{s \in s_\infty(T)} \text{weight}(s),$$

where $s_0(T)$ and $s_\infty(T)$ are the sets of states whose boundary arcs have connectivity type T_0 and T_∞, respectively. Then

$$\langle T \rangle = f_T(A)\,\langle T_0 \rangle + g_T(A)\,\langle T_\infty \rangle.$$

The pair $\mathbf{br}(T) := (f_T(A), g_T(A))$ is called the *bracket vector* of T.

Proof. By the state-sum definition of the Kauffman bracket [2],

$$\langle T \rangle = \sum_s \text{weight}(s) = \sum_{s \in s_0(T)} \text{weight}(s) + \sum_{s \in s_\infty(T)} \text{weight}(s).$$

For a state $s \in s_0(T)$, the boundary arcs have connectivity type T_0. If we close T up along its boundary in the pattern of T_0, the resulting diagram is T_0 together with some number of disjoint circles; each circle contributes a factor $(-A^2 - A^{-2})$ to the bracket. Thus all states in $s_0(T)$ contribute a common factor $\langle T_0 \rangle$, and we can write

$$\sum_{s \in s_0(T)} \text{weight}(s) = f_T(A)\,\langle T_0 \rangle.$$

The same reasoning applied to $s_\infty(T)$ yields

$$\sum_{s\in s_\infty(T)} \mathrm{weight}(s) = g_T(A)\,\langle T_\infty\rangle.$$

Adding these two expressions gives the desired decomposition. □

3.3. **Rotation Invariance.**

Lemma 3.2 (Rotation invariance)**.** Let ρ be a rotation of the 3–ball containing a 2–tangle T through angle π about any axis that preserves the four boundary points. Then

$$\mathbf{br}(\rho(T)) = \mathbf{br}(T).$$

Proof. Each state s of T corresponds to a state $\rho(s)$ of $\rho(T)$, obtained by making the same smoothing choice at corresponding crossings. A π–rotation preserves

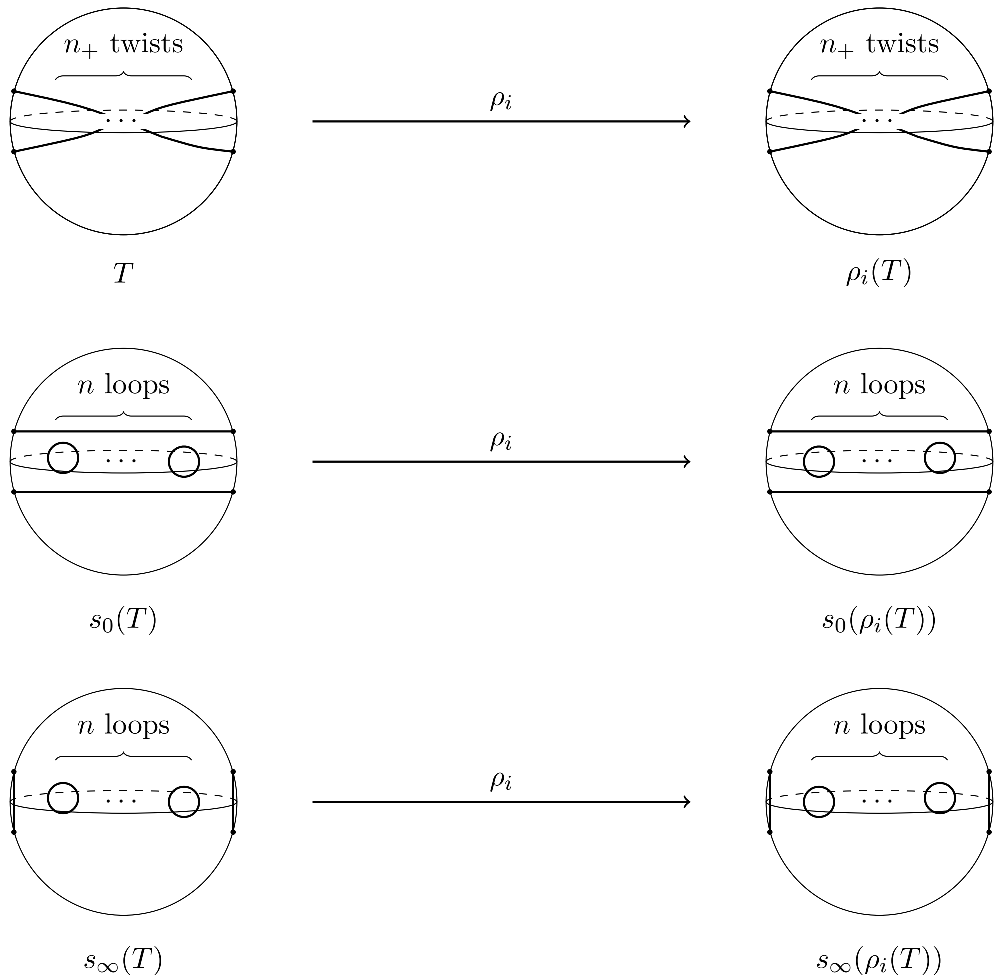
$$a(\rho(s)) = a(s),\quad b(\rho(s)) = b(s),\quad \ell(\rho(s)) = \ell(s),$$

so $\mathrm{weight}(\rho(s)) = \mathrm{weight}(s)$ (cf. Kauffman's state model [2] and the tangle symmetry discussion in Lickorish [3, §8]).

The connectivity type of the boundary arcs is preserved as well, so

$$\rho : s_0(T) \to s_0(\rho(T)),\qquad \rho : s_\infty(T) \to s_\infty(\rho(T)),$$

and



are bijections.

Thus

$$f_{\rho(T)}(A) = \sum_{s'\in s_0(\rho(T))} \mathrm{weight}(s') = \sum_{s\in s_0(T)} \mathrm{weight}(s) = f_T(A),$$

and the same for $g_T(A)$, giving $\mathbf{br}(\rho(T)) = \mathbf{br}(T)$. □

4. Kauffman Bracket Polynomial under Mutation

Lemma 4.1. Let $K = F \cup G$ be a decomposition of a knot into two 2–tangles F and G along a Conway sphere. There exist polynomials $\alpha(A), \beta(A) \in \mathbb{Z}[A, A^{-1}]$, depending only on G, such that

$$\langle K \rangle = f_F(A)\,\alpha(A) + g_F(A)\,\beta(A),$$

where $\mathbf{br}(F) = (f_F, g_F)$ is the bracket vector of F.

Proof. Let $\mathcal{S}$ be the skein module of 2–tangles. Gluing the fixed outside tangle G to a 2–tangle T and closing up along the Conway sphere gives a link diagram $G \cup T$. The map

$$\Phi_G : \mathcal{S} \to \mathbb{Z}[A, A^{-1}], \qquad T \mapsto \langle G \cup T \rangle$$

is $\mathbb{Z}[A, A^{-1}]$–linear, since the Kauffman bracket respects the skein relations [2, 3].

Set

$$\alpha(A) := \Phi_G(T_0) = \langle G \cup T_0 \rangle, \qquad \beta(A) := \Phi_G(T_\infty) = \langle G \cup T_\infty \rangle.$$

By Lemma 3.1, any 2–tangle F satisfies

$$\langle F \rangle = f_F \langle T_0 \rangle + g_F \langle T_\infty \rangle,$$

so by linearity of Φ_G we obtain

$$\langle K \rangle = \langle G \cup F \rangle = f_F\,\langle G \cup T_0 \rangle + g_F\,\langle G \cup T_\infty \rangle = f_F\,\alpha + g_F\,\beta.$$

□

Lemma 4.2. Let $K = F \cup G$ and $K' = F' \cup G$ be mutant knots, where $F' = \rho(F)$ for a π–rotation ρ as in the definition of mutation. Then

$$\langle K \rangle = \langle K' \rangle.$$

Proof. By Lemma 4.1,

$$\langle K \rangle = f_F\,\alpha + g_F\,\beta, \qquad \langle K' \rangle = f_{F'}\,\alpha + g_{F'}\,\beta,$$

for some α, β depending only on G. By Lemma 3.2, $\mathbf{br}(F') = \mathbf{br}(F)$, so $f_{F'} = f_F$ and $g_{F'} = g_F$. Hence the two linear combinations are equal and $\langle K \rangle = \langle K' \rangle$, a standard fact in mutation theory [3]. □

5. Jones Polynomial Invariance Under Mutation

5.1. Writhe and mutation.

Lemma 5.1. Let D be an oriented diagram of a knot, written as $D = F \cup G$ where F is the 2–tangle inside a Conway sphere and G is the outside tangle. Let $D' = F' \cup G$ be the diagram obtained by replacing F with its π–rotation $F' = \rho_i(F)$ for one of the three rotations ρ_1, ρ_2, ρ_3. Then

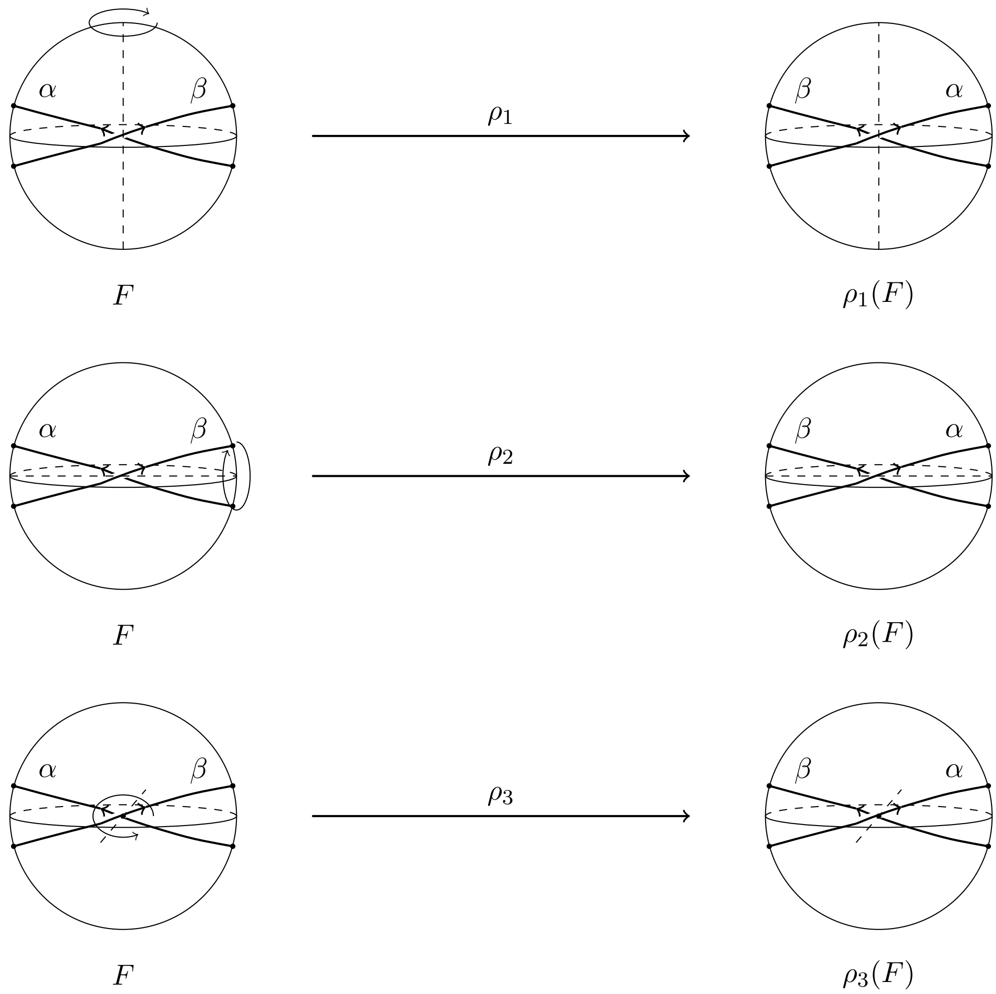
$$w(D) = w(D').$$

Proof. Since the outside tangle G is unchanged, all crossings outside the Conway sphere contribute equally to $w(D)$ and $w(D')$; this follows directly from the definition of writhe [2, 3].

Inside the sphere, the tangle F contains a single oriented crossing, formed by two oriented arcs which we denote by α and β. Under any π–rotation ρ_i, these two arcs are carried to their corresponding arcs in F' with their orientations preserved. The over–strand remains the over–strand, the under–strand remains the under–strand, and the directed tangent vectors at the crossing point are rotated consistently. Thus the oriented local picture of the crossing is unchanged up to rigid motion, as discussed in the mutation framework of [3, 7].

In particular, the sign of the crossing is preserved,

$$\mathrm{sgn}(\text{crossing in } F) = \mathrm{sgn}(\text{corresponding crossing in } F' = \rho_i(F)).$$



Therefore the total contribution to the writhe from crossings inside the sphere is the same in D and D', and combining this with the outside contribution gives

$$w(D) = w(D').$$

□

5.2. **Invariance of the Jones polynomial.**

Theorem 5.1. If K and K' are related by Conway mutation, then

$$V_K(t) = V_{K'}(t).$$

Proof. Choose oriented diagrams D and D' for K and K' as in Lemma 5.1. By Lemma 4.2 (see also [4, 7]),

$$\langle D\rangle = \langle D'\rangle,$$

and by Lemma 5.1 (cf. the orientation conventions in [2, 3]),

$$w(D) = w(D').$$

Thus, using the normalized bracket definition of the Jones polynomial [1, 2, 3],

$$f_D(A) = (-A^3)^{-w(D)}\langle D\rangle = (-A^3)^{-w(D')}\langle D'\rangle = f_{D'}(A).$$

Evaluating at $A = t^{-1/4}$ (as in [1, 8, 6]) gives

$$V_K(t) = f_D\big(t^{-1/4}\big) = f_{D'}\big(t^{-1/4}\big) = V_{K'}(t).$$

□

6. Mutation of the Unknot and the Jones–Unknot Problem

6.1. Mutation of the unknot.

Definition 6.1. A knot K has *trivial Jones polynomial* if

$$V_K(t) = 1.$$

The unknot $\bigcirc$ satisfies $V_{\bigcirc}(t) = 1$ [1, 2, 3].

Theorem 6.1 (Mutation preserves the unknot)**.** If K is the unknot, then any mutant of K is again the unknot (see [4, 7] for background on mutation and Jones polynomials).

6.2. **The Jones–Unknot Problem.** The Jones polynomial assigns to each knot K a Laurent polynomial $V_K(t) \in \mathbb{Z}[t^{\pm 1/2}]$ obtained from the Kauffman bracket by the standard normalization [1, 2, 3]. For the unknot $\bigcirc$ one has the familiar value

$$V_{\bigcirc}(t) = 1.$$

A natural question, still open after forty years, is whether the Jones polynomial detects the unknot. That is

If $V_K(t) = V_{\bigcirc}(t)$, must K be the unknot?

This is known as the *Jones–unknot problem.* No nontrivial knot is known to share the Jones polynomial of the unknot, and no proof is known that such a knot cannot exist (see discussions in [6, 8]).

7. Conclusion

We have seen that Conway mutation preserves the Jones polynomial through two independent mechanisms. The Kauffman bracket of the inside tangle remains unchanged under any π–rotation of the Conway sphere, and the writhe is likewise preserved after adjusting orientations. As a result,

$$(-A^3)^{-w(K)}\langle K\rangle = (-A^3)^{-w(K')}\langle K'\rangle, \qquad V_K(t) = V_{K'}(t),$$

so mutant knots are always Jones–equivalent.

In particular, classical examples such as the Conway and Kinoshita–Terasaka knots illustrate the limitations of the Jones polynomial as a complete knot invariant: they are topologically distinct but algebraically identical from the viewpoint of $V_K(t)$. Nevertheless, the Jones polynomial still detects many subtle features of knot structure, and understanding its behavior under mutation continues to provide important insight into the algebraic and geometric aspects of knots and tangles.

References

[1] V. F. R. Jones, A polynomial invariant for knots via von Neumann algebras, *Bull. Amer. Math. Soc. (N.S.)* **12** (1985), 103–111.

[2] L. H. Kauffman, State models and the Jones polynomial, *Topology* **26** (1987), 395–407.

[3] W. B. R. Lickorish, *An Introduction to Knot Theory*, Graduate Texts in Mathematics, vol. 175, Springer, 1997.

[4] H. R. Morton, Mutant knots, chapter in *Handbook of Knot Theory*, Elsevier, 2005.

[5] L. H. Kauffman, *On Knots*, Annals of Mathematics Studies, vol. 115, Princeton University Press, 1987.

[6] C. C. Adams, *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, American Mathematical Society, 2004.

[7] P. R. Cromwell, *Knots and Links*, Cambridge University Press, 2004.

[8] A. Kawauchi, *A Survey of Knot Theory*, Birkhäuser, 1996.

Department of Mathematics, Columbia University

Email address: ilt2109@columbia.edu