Classical vs Path Coupling on a Geometric State Space: Glauber Dynamics for q-Colorings of C_4

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1. Setup and geometry: C_4, state space $\Omega_q(C_4)$, Hamming metric
2. Chain: single-site Glauber dynamics and uniform stationarity
3. Two tools: classical coupling vs path coupling
4. Results: constants in linear-in-$(q-1)$ mixing bounds
5. Computations: TV decay and semilog diagnostic

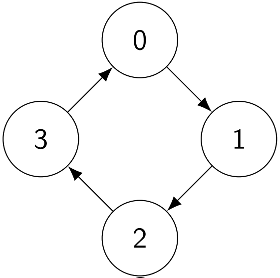
We study proper q-colorings of the four-cycle

$$V(C_4) = \{0, 1, 2, 3\}, \qquad E = \{(i, i+1 \bmod 4)\}.$$

Each vertex has degree 2, so a recoloring step forbids at most two colors.

Useful rule of thumb (appears in both couplings)

$|L_x(v)| \geq q - 2$, legal-set overlaps are typically large.



C_4

Definition (Proper coloring)

A **proper** q**-coloring** is a map $x : V \to [q]$ such that

$$x(i) \neq x(j) \quad \text{whenever} \quad (i,j) \in E.$$

State space and size

$$\Omega_q(C_4) = \{x : \text{proper}\}, \qquad |\Omega_q(C_4)| = q(q-1)(q-2)^2.$$

Geometry on Ω: embed $\Omega \subset [q]^4$ and use Hamming distance

$$d_H(x,y) = |\{i \in V : \ x(i) \neq y(i)\}| \leq 4.$$

Adjacency $x \sim y$ means $d_H(x,y) = 1$ (a single-vertex change).

At each step:

1. choose $v \in V$ uniformly
2. recolor v uniformly from the legal set

$$L_x(v) = \{c \in [q] : c \neq x(u)\ \forall u \sim v\}.$$

Uniform stationarity (by symmetry of recolor moves)

$$\pi(x) = \frac{1}{|\Omega_q(C_4)|}.$$

Geometric intuition: each move "change one coordinate to a legal color" has a reverse move of the same probability.

Definition (TV distance and mixing time)

$$d_x(t) = \|P^t(x,\cdot) - \pi\|_{\mathrm{TV}}, \qquad t_{\mathrm{mix}}(\varepsilon) = \min\{t : \max_{x\in\Omega} d_x(t) \le \varepsilon\}.$$

Lemma (Coupling inequality)

For any coupling (X_t, Y_t) *and* $\tau = \min\{t : X_t = Y_t\}$,

$$\|P^t(x,\cdot) - P^t(y,\cdot)\|_{\mathrm{TV}} \le \mathrm{P}_{x,y}(\tau > t).$$

Two strategies: classical coupling bounds coalescence directly; path coupling contracts a metric on Hamming edges.

Couple two copies (X_t, Y_t) as follows:

- update the same vertex v in both chains
- maximal coupling of recolor choices from $L_X(v)$ and $L_Y(v)$

Track the global disagreement statistic

$$D_t = \left|\{i \in V : \ X_t(i) \neq Y_t(i)\}\right| \in \{0, 1, 2, 3, 4\}.$$

Geometric input (degree 2):

$$|L_X(v)|, |L_Y(v)| \geq q - 2, \quad |L_X(v) \cap L_Y(v)| \text{ is often } \geq q - 3,$$

so the "failure-to-match" probability is $O(1/(q-1))$.

When v is a disagreement site, a maximal coupling forces agreement with probability at least

$$\frac{q-3}{q-2}.$$

Since v is chosen with probability $1/4$, this creates a uniform negative drift in D_t.

Lemma (Classical coupling scale)

For $q \geq 3$,

$$\mathrm{E}[\tau_{\mathrm{cl}}] \leq 6\,(q-1).$$

Talking point: the constant is a finite-state drift computation plus the small geometry of C_4.

Equip Ω with Hamming distance d_H.

Path coupling philosophy:

- only analyze adjacent pairs $x \sim y$ with $d_H(x, y) = 1$
- prove one-step contraction $\mathrm{E}[d_H(X_1, Y_1)] \leq \alpha(q) d_H(x, y)$
- extend to all pairs by chaining along a Hamming path

Geometry statement: contraction on edges of the metric graph induced by $\Omega \subset [q]^4$ drives mixing.

Take $x \sim y$, differing only at one vertex v. Couple one step by: same updated vertex u, then maximal coupling on recolor choices.

Lemma (Edge contraction)

For all $q \geq 3$,

$$\mathrm{E}[d_H(X_1, Y_1)] \leq \left(1 - \frac{1}{4(q-1)}\right) d_H(x, y).$$

Equivalently, $\alpha(q) = 1 - \frac{1}{4(q-1)}$.

Heuristic: the disagreeing vertex is hit with probability $1/4$ and then coalesces with probability $\gtrsim 1/(q-1)$.

With $1-\alpha(q)=\frac{1}{4(q-1)}$, Bubley–Dyer yields

$$t_{\mathrm{mix}}^{\mathrm{pc}}(\varepsilon) \;\le\; \frac{\log(d_{\max})+\log(\varepsilon^{-1})}{1-\alpha(q)}.$$

On C_4 we have $d_{\max}\le 4$. A convenient "paper-compatible" form is

$$t_{\mathrm{mix}}^{\mathrm{pc}}(\varepsilon)\le 4\,(q-1)\big(\log|\Omega|+\log(\varepsilon^{-1})\big), \qquad |\Omega|=q(q-1)(q-2)^2.$$

Emphasis: the bound is local-to-global via the Hamming-path geometry.

Both bounds take the same form

$$t_{\mathrm{mix}}(\varepsilon) \;\lesssim\; C\,(q-1)\big(\log|\Omega| + \log(\varepsilon^{-1})\big).$$

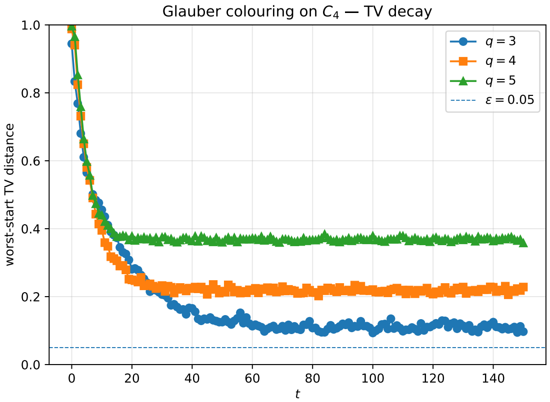
Classical coupling gives $C = 6$.

Path coupling gives $C = 4$.

Statement (Derived bound separation)

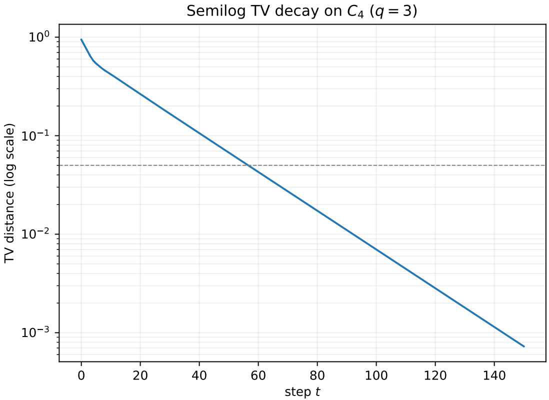
For all $q \geq 3$ and $\varepsilon \in (0, 1/2]$,

$$t_{\mathrm{mix}}^{\mathrm{pc}}(\varepsilon) \;<\; t_{\mathrm{mix}}^{\mathrm{cl}}(\varepsilon).$$



10^3 trajectories per start state; dashed line at $\varepsilon = 0.05$.

On a semilog scale, approximate linearity is a strong diagnostic for exponential decay.



Main takeaways

- On C_4, both methods give linear-in-$(q-1)$ mixing bounds with explicit constants.
- Path coupling wins here because Hamming-edge contraction is easy and strong.
- The "geometry" is twofold: degree-2 constraints on C_4 and Hamming-path structure in Ω.

Questions?

R. Bubley and M. Dyer.
Path coupling: a technique for proving rapid mixing in Markov chains.
In *Proc. 38th IEEE FOCS*, 1997.

D. A. Levin, Y. Peres, and E. L. Wilmer.
Markov Chains and Mixing Times.
AMS, 2nd ed., 2017.

M. Jerrum.
A very simple algorithm for estimating the number of k-colorings of a low-degree graph.
Random Structures & Algorithms, 1995.

M. Dyer, A. Frieze, and M. Jerrum.
On the rapid mixing of Glauber dynamics for graph colorings.