Conway Mutation and the Jones Polynomial

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Knot Theory – Final Project

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Preliminaries

Kauffman bracket. For an unoriented diagram D,

$$\langle \varnothing \rangle = 1, \qquad \langle D \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\,\langle D \rangle,$$

and at a crossing

$$\left\langle \times \right\rangle = A \left\langle \asymp \right\rangle + A^{-1} \left\langle)(\right\rangle$$
$$\left\langle \times \right\rangle = A \left\langle)(\right\rangle + A^{-1} \left\langle \asymp \right\rangle$$

Writhe. For an oriented D,

$$w(D) = \sum_c \mathrm{sgn}(c) \quad (\text{sum of } \pm 1 \text{ over crossings})$$

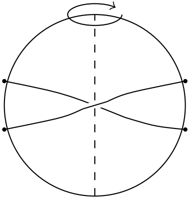
Jones polynomial. For a knot K with oriented diagram D,

$$f_D(A) = (-A^3)^{-w(D)} \langle D \rangle, \qquad V_K(t) = f_D\big(t^{-1/4}\big)$$

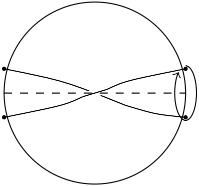
The Three π–Rotations of a Conway Sphere

Conway sphere. An embedded 2–sphere S in $\mathbb{R}^3$ meeting a knot K in 4 points. The sphere S separates space into an inside/outside region. It cuts K into two 2–tangles

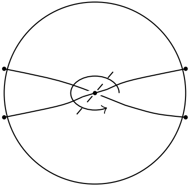
$$K = F \cup G$$



$\rho_1(F)$



$\rho_2(F)$



$\rho_3(F)$

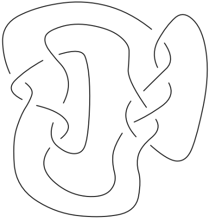
Conway mutation. Let ρ_i be one of the three π–rotations of the 3–ball containing F that fixes the four boundary points. Set

$$F' = \rho_i(F), \qquad K' = F' \cup G$$

The knot K' is called a *mutant* of K.

Example: Conway Knot C and Kinoshita–Terasaka Knot KT

$C =$



$KT =$

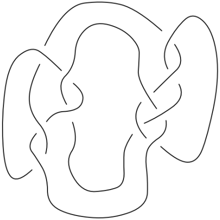


Figure: The Conway and Kinoshita-Terasaka knots, as shown in Morton [4], are a classic example of inequivalent mutant knots, sharing the same Jones polynomial.

Jones polynomials (same, this is canonical).

$$V_C(t) = V_{KT}(t) = t^{-4}\left(-1 + 2t - 2t^2 + 2t^3 + t^6 - 2t^7 + 2t^8 - 2t^9 + t^{10}\right)$$

Key Lemmas Behind the Invariance

Lemma 1 (Bracket vector for a 2–tangle)
For any 2–tangle T there are unique polynomials $f_T(A), g_T(A) \in \mathbb{Z}[A, A^{-1}]$ such that

$$\langle T \rangle = f_T(A) \left\langle \asymp \right\rangle + g_T(A) \left\langle (\ \ ) \right\rangle$$

We write

$$\mathrm{br}(T) := (f_T(A), g_T(A))$$

Lemma 2 (Rotation invariance)
If ρ is a π–rotation of the ball containing T that preserves the 4 boundary points, then

$$\mathrm{br}(\rho(T)) = \mathrm{br}(T)$$

Intuition: every state of T becomes a state of $\rho(T)$ with the same number of loops and the same $0/\infty$ tangle, so the two coefficients f_T, g_T are unchanged.

Proof Sketch: Jones Polynomial is Invariant

Let

$$K = F \cup G, \qquad K' = F' \cup G, \qquad F' = \rho(F)$$

be a Conway mutant pair.

Lemma 3 (Outside tangle is linear)
There are polynomials $\alpha(A), \beta(A)$ depending only on G such that

$$\langle K \rangle = f_F(A)\,\alpha(A) + g_F(A)\,\beta(A), \qquad \langle K' \rangle = f_{F'}(A)\,\alpha(A) + g_{F'}(A)\,\beta(A)$$

Step 1. Kauffman bracket
By Lemma 2 (Rotation invariance), $\mathrm{br}(F') = \mathrm{br}(F)$, so

$$(f_{F'}, g_{F'}) = (f_F, g_F) \quad \Longrightarrow \quad \langle K \rangle = \langle K' \rangle$$

Step 2. Writhe
Rotating the tangle by π and adjusting orientations preserves the sign of every crossing, so

$$w(K) = w(K')$$

Conclusion

Therefore. The normalized bracket is the same for K and K'

$$(-A^3)^{-w(K)}\langle K\rangle = (-A^3)^{-w(K')}\langle K'\rangle$$

so their Jones polynomials agree

$$V_K(t) = V_{K'}(t)$$

Unknot and the Jones–unknot problem.

- The Jones polynomial of the unknot is $V_{\bigcirc}(t) = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$.
- It is unknown whether $V_K(t) = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$ forces K to be the unknot (the Jones–unknot problem).

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