A Quantitative Modulus Control on Annuli

Iz Tecúm

Columbia University

Spring 2026

Goal of this talk (pick a few theorems, prove them carefully)

Annuli are the first non-simply-connected planar domains. Their conformal geometry has exactly one continuous parameter:

$$\operatorname{mod}\left(A(r,R)\right) = \frac{1}{2\pi}\log\left(\frac{R}{r}\right).$$

We prove (with explicit constants) a small set of "dictionary entries" showing how $\operatorname{mod}(A)$ controls:

- variational geometry (extremal length, capacity),
- analytic structure (strip lift $\Rightarrow$ Fourier/Laurent decay),
- intrinsic geometry (hyperbolic density on $A(r,R)$, core geodesic length),
- mapping constraints (Schwarz–Pick on annuli, classification of proper maps).

Two models you should keep in your head

Round model.
$A(r,R) = \{z \in \mathbb{C} : \; r < |z| < R\}$.

- Boundary components: $|z| = r$ and $|z| = R$.
- Core circle: $|z| = \sqrt{rR}$.

Strip/cylinder model. Let $w = \log z$.

$$S_{r,R} = \{\log r < \Re w < \log R\}, \qquad w \sim w + 2\pi i.$$

$$W := \log(R/r) = 2\pi \;\mathrm{mod}(A(r,R))$$

is the *strip width* and the ambient scale for every estimate.

Outline

Classification (one real parameter)

Theorem 1 (Conformal classification of annuli)

Every doubly connected domain $A \subset \mathbb{C}$ with nondegenerate boundary components is conformally equivalent to a round annulus $A(r, R)$, uniquely up to scaling. The invariant is

$$\operatorname{mod}(A) = \frac{1}{2\pi} \log(R/r).$$

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Consequence. Any conformally natural scalar functional on annuli can only depend on $\operatorname{mod}(A)$.

Lift mechanism: holomorphic analysis on a periodic strip

If $f \in \mathrm{Hol}(A(r,R), Y)$, define the lift

$$F(w) := f(e^w) \qquad (w \in S_{r,R}),$$

so F is holomorphic on the strip and $2\pi i$-periodic:

$$F(w + 2\pi i) = F(w).$$

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Hence annulus questions $\iff$ periodic strip questions on width $W = \log(R/r)$. This is where $\mathrm{mod}(A)$ enters: it is exactly $W/(2\pi)$.

Outline

Canonical curve families on an annulus

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These are dual families: one crosses the cylinder, the other wraps around it.

Original statement 1: modulus as extremal length

Theorem 2 (Extremal length identities)

For an annulus A,

$$\mathrm{Ext}_A(\Gamma_{\mathrm{rad}}) = \mathrm{mod}(A), \qquad \mathrm{Ext}_A(\Gamma_{\mathrm{sep}}) = \frac{1}{\mathrm{mod}(A)}, \qquad \mathrm{Ext}_A(\Gamma_{\mathrm{rad}})\,\mathrm{Ext}_A(\Gamma_{\mathrm{sep}}) = 1.$$

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Area.

$$\int_{A(r,R)} \rho_*^2 \, dA = \int_0^{2\pi} \int_r^R \frac{1}{\rho^2} \, \rho \, d\rho \, d\theta = 2\pi \log(R/r) = 2\pi W.$$

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Thus

$$\mathrm{Ext}(\Gamma_{\mathrm{rad}}) = \sup_\rho \frac{\left(\inf_{\gamma \in \Gamma_{\mathrm{rad}}} \int_\gamma \rho |dz|\right)^2}{\int \rho^2 \, dA} \geq \frac{W^2}{2\pi W} = \frac{W}{2\pi} = \mathrm{mod}(A(r, R)).$$

Proof II (sharpness): reduce to a rectangle upstairs

Under $w = \log z$, the metric $\rho_*(z) = 1/|z|$ becomes *constant* upstairs:

$$|dz| = |e^w||dw| \;\Rightarrow\; \rho_*(z)|dz| = \frac{1}{|e^w|}|e^w||dw| = |dw|.$$

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So Γ_{rad} lifts to curves crossing a rectangle of width W and height 2π. The extremal length of the left-to-right crossing family in a Euclidean rectangle is

$$\frac{\text{width}}{\text{height}} = \frac{W}{2\pi} = \mathrm{mod}(A(r,R)),$$

so the lower bound is sharp. The separating family is dual, giving the reciprocal identity.

Original statement 2: capacity of a round annulus

Theorem 3 (Capacity formula)

For $A(r, R)$, the condenser capacity between the boundary circles is

$$\operatorname{cap}(A(r,R)) = \frac{2\pi}{\log(R/r)} = \frac{1}{\operatorname{mod}(A(r,R))}.$$

An energy-minimizing harmonic function with boundary values 0 *on* $|z| = r$ *and* 1 *on* $|z| = R$ *is*

$$u(z) = \frac{\log|z| - \log r}{\log R - \log r}.$$

Proof sketch: why the minimizer is logarithmic

If u depends only on $\rho = |z|$, the Laplacian in polar coordinates gives

$$\Delta u = u_{\rho\rho} + \frac{1}{\rho} u_\rho = 0 \quad \Longrightarrow \quad (\rho u_\rho)' = 0 \quad \Longrightarrow \quad u(\rho) = a \log \rho + b.$$

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Impose $u(r) = 0$ and $u(R) = 1$ to obtain the formula. Then compute the Dirichlet energy:

$$\int_{A(r,R)} |\nabla u|^2 \, dA = \int_0^{2\pi} \int_r^R \left| \frac{d}{d\rho} \left(\frac{\log \rho - \log r}{\log R - \log r} \right) \right|^2 \rho \, d\rho \, d\theta = \frac{2\pi}{\log(R/r)}.$$

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Uniqueness follows from strict convexity of the Dirichlet energy on the affine class of admissible boundary data.

Outline

Hyperbolic metric: why it matters here

For a hyperbolic Riemann surface Ω (in particular, any annulus), there is a unique complete conformal metric

$$ds_\Omega = \rho_\Omega(z)\,|dz|, \qquad K_\Omega \equiv -1.$$

If $f \in \mathrm{Hol}(\Omega, \mathbb{D})$, Schwarz–Pick in metric form is

$$\rho_{\mathbb{D}}(f(z))\,|f'(z)| \leq \rho_\Omega(z).$$

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So *explicit control of ρ_A on an annulus* gives explicit derivative bounds for all disk-valued holomorphic maps.

Original statement 3: explicit hyperbolic density on a round annulus

Let $A = A(r, R)$. Set the strip width $W = \log(R/r)$.

Theorem 4 (Hyperbolic density on a round annulus)

For $z \in A(r, R)$,

$$\rho_A(z) = \frac{\pi}{W} \cdot \frac{1}{|z|} \csc\left(\frac{\pi}{W} \log\left(\frac{|z|}{r}\right)\right).$$

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Key features:

- $\rho_A(z) \to \infty$ as $|z| \downarrow r$ or $|z| \uparrow R$ (metric completeness).
- The only global scale is the prefactor $\pi/W = \frac{1}{2\,\mathrm{mod}(A)}$.
- Radial symmetry: ρ_A depends only on $|z|$ in the round model.

Proof I: reduce to the strip and use a standard formula

Lift by $z = e^w$ with $w = x + iy$. The strip model is

$$S = \{x \in (\log r, \log R)\}, \qquad w \sim w + 2\pi i.$$

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The hyperbolic density on a (non-periodic) strip of width W,

$$\widetilde{S} = \{0 < x < W\},$$

is explicitly

$$\rho_{\widetilde{S}}(w) = \frac{\pi}{W} \csc\Big(\frac{\pi x}{W}\Big).$$

(One way: map $\widetilde{S}$ conformally to $\mathbb{D}$ via $w \mapsto \exp(\pi i w/W)$ and pull back the disk metric.)

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(One way: map $\widetilde{S}$ conformally to $\mathbb{D}$ via $w \mapsto \exp(\pi i w/W)$ and pull back the disk metric.) Because the annulus quotient is by vertical translation (an isometry of $\widetilde{S}$), the density descends to the cylinder.

Proof II: push the strip density down to the annulus

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Therefore

$$\rho_A(z)\,|dz| = \rho_{\widetilde{S}}(w)\,|dw| = \frac{\pi}{W}\,\csc\left(\frac{\pi(x - \log r)}{W}\right) \cdot \frac{|dz|}{|z|}.$$

Cancel $|dz|$ and substitute $x - \log r = \log(|z|/r)$ to obtain

$$\rho_A(z) = \frac{\pi}{W} \cdot \frac{1}{|z|}\,\csc\left(\frac{\pi}{W}\log\left(\frac{|z|}{r}\right)\right).$$

Core geodesic length (a clean modulus-dependent quantity)

In the cylinder picture, the unique closed geodesic in the nontrivial homotopy class is the "core" circle. In the round model this is $C_* = \{|z| = \sqrt{rR}\}$.

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In the cylinder picture, the unique closed geodesic in the nontrivial homotopy class is the "core" circle. In the round model this is $C_* = \{|z| = \sqrt{rR}\}$. [Core geodesic length] Let $A = A(r, R)$ and $W = \log(R/r)$. Then the hyperbolic length of the core closed geodesic satisfies

$$\ell_A(C_*) = \int_0^{2\pi} \rho_A\big(\sqrt{rR}\, e^{i\theta}\big) \cdot \left| \tfrac{d}{d\theta}\big(\sqrt{rR}\, e^{i\theta}\big) \right| d\theta = \frac{2\pi^2}{W} = \frac{\pi}{\mathrm{mod}(A)}.$$

Proof: plug the core radius into the explicit density

At $|z| = \sqrt{rR}$, we have

$$\log\left(\frac{|z|}{r}\right) = \log\left(\sqrt{\frac{R}{r}}\right) = \frac{1}{2}\log(R/r) = \frac{W}{2}.$$

Hence

$$\csc\left(\frac{\pi}{W}\log(|z|/r)\right) = \csc(\pi/2) = 1.$$

So

$$\rho_A(z) = \frac{\pi}{W}\cdot\frac{1}{|z|}.$$

The Euclidean speed along the circle is $|dz| = |z|\,d\theta$. Therefore

$$\ell_A(C_*) = \int_0^{2\pi}\frac{\pi}{W}\cdot\frac{1}{|z|}\cdot|z|\,d\theta = \frac{\pi}{W}\cdot 2\pi = \frac{2\pi^2}{W} = \frac{\pi}{\operatorname{mod}(A)}.$$

Outline

Schwarz–Pick (metric form) specialized to annuli

For $f \in \mathrm{Hol}(A, \mathbb{D})$,

$$\rho_{\mathbb{D}}(f(z))\,|f'(z)| \le \rho_A(z).$$

Since $\rho_{\mathbb{D}}(w) = \frac{2}{1-|w|^2}$, we obtain the pointwise derivative bound

$$|f'(z)| \le \frac{1-|f(z)|^2}{2}\,\rho_A(z) \;\le\; \frac{1}{2}\rho_A(z).$$

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So on a round annulus $A(r, R)$,

$$|f'(z)| \leq \frac{1}{2} \cdot \frac{\pi}{W} \cdot \frac{1}{|z|}\,\csc\left(\frac{\pi}{W}\log\left(\frac{|z|}{r}\right)\right).$$

Uniform interior bounds on the ε-core

Define the ε-core (in logarithmic coordinates)

$$A_\varepsilon = \left\{ z \in A(r,R) : \ \varepsilon \le \log\left(\frac{|z|}{r}\right) \le W - \varepsilon \right\}.$$

On A_ε, the sine term is bounded away from 0:

$$\sin\left(\frac{\pi}{W}\log(|z|/r)\right) \ge \sin\left(\frac{\pi\varepsilon}{W}\right).$$

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$$\sin\left(\frac{\pi}{W}\log(|z|/r)\right) \ge \sin\left(\frac{\pi\varepsilon}{W}\right).$$

Therefore, for all $z \in A_\varepsilon$ and all $f \in \mathrm{Hol}(A, \mathbb{D})$,

$$|f'(z)| \le \frac{1}{2}\rho_A(z) \le \frac{1}{2} \cdot \frac{\pi}{W} \cdot \frac{1}{|z|} \csc\left(\frac{\pi\varepsilon}{W}\right).$$

This makes the modulus dependence explicit through $W = 2\pi \operatorname{mod}(A)$.

Outline

Original statement 4: strip width suppresses nonzero modes

Let $A = A(r, R)$ with $W = \log(R/r)$. Let f be holomorphic on A and bounded: $\|f\|_\infty \le M$. Write its Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n.$$

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Theorem 5 (Mode suppression on long cylinders)

There is an explicit constant C (depending only on normalization choices) such that

$$|a_n| \leq C\, M\, e^{-|n|W/2} \qquad (n \in \mathbb{Z}).$$

In particular, as $\operatorname{mod}(A) \to \infty$ (i.e. $W \to \infty$), all nonconstant modes are exponentially damped.

Proof idea: periodic strip Fourier series

Lift f to the strip: $F(w) = f(e^w)$, so F is holomorphic and $2\pi i$-periodic. Thus F has a Fourier series in the vertical direction:

$$F(w) = \sum_{n \in \mathbb{Z}} c_n e^{nw} \qquad \text{(since } e^{n(w+2\pi i)} = e^{nw}\text{).}$$

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But $e^{nw} = e^{n(\log z)} = z^n$, so $c_n = a_n$. Hence controlling Fourier coefficients of F on a strip of width W gives bounds on Laurent coefficients of f.

Proof sketch: Cauchy estimate across the strip

Fix $n \geq 1$. Consider

$$a_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{n+1}} \, dz$$

for any $\rho \in (r, R)$. Then

$$|a_n| \leq \frac{1}{2\pi} \cdot (2\pi\rho) \cdot \frac{M}{\rho^{n+1}} = \frac{M}{\rho^n}.$$

Similarly, for $n \leq -1$,

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Optimize by choosing $\rho = \sqrt{rR}$ (the geometric mean), giving

$$|a_n| \leq M \left(\sqrt{\frac{r}{R}}\right)^n = Me^{-nW/2} \qquad (n \geq 1),$$

and

$$|a_n| \leq M \left(\sqrt{\frac{r}{R}}\right)^{|n|} = Me^{-|n|W/2} \qquad (n \leq -1).$$

This is the exponential damping with rate $W/2 = \pi \operatorname{mod}(A)$.

Outline

Original statement 5: proper maps are monomials

Let $A = A(r, R)$ and $A' = A(r', R')$ be round annuli.

Theorem 6 (Classification of proper holomorphic maps)

If $f : A \to A'$ is a proper holomorphic map of degree $n \in$, then after composing with rotations (and possibly inversion),

$$f(z) = c\,z^{\pm n} \qquad (c \in \mathbb{C} \setminus \{0\}).$$

Moreover the moduli scale by degree:

$$\mathrm{mod}(A') = n\ \mathrm{mod}(A).$$

Proof I: lift to strips and use periodicity

Lift to strips via $z = e^w$ and $z' = e^{w'}$. A holomorphic map $f : A \to A'$ lifts to a holomorphic map

$$F : S_{r,R} \to S_{r',R'}$$

satisfying the deck compatibility

$$F(w + 2\pi i) = F(w) + 2\pi i\, m$$

for some integer m (the induced map on $\pi_1(A) \cong \mathbb{Z}$).

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for some integer m (the induced map on $\pi_1(A) \cong \mathbb{Z}$). Properness and degree n force $m = \pm n$. Hence

$$G(w) := F(w) - mw$$

is $2\pi i$-periodic and holomorphic on the strip.

Proof II: a bounded periodic holomorphic function is constant

Because F maps a vertical strip to a vertical strip, the real part $\Re F(w)$ stays in $(\log r', \log R')$. Thus

$$\Re G(w) = \Re F(w) - m\,\Re w$$

is bounded on the strip (since $\Re w$ is confined to a bounded interval of length W).

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is bounded on the strip (since $\Re w$ is confined to a bounded interval of length W). A holomorphic function on a strip with bounded real part is constant (apply the maximum principle to e^G or use harmonicity of $\Re G$ and periodicity). Therefore $G(w) \equiv \beta$ for some constant $\beta \in \mathbb{C}$.

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$$F(w) = mw + \beta.$$

Exponentiating gives

$$f(e^w) = e^{F(w)} = e^{\beta} e^{mw} = c\,z^m, \qquad c = e^{\beta}.$$

Thus $f(z) = cz^{\pm n}$.

Modulus scaling: why $\operatorname{mod}(A') = n \operatorname{mod}(A)$

If $F(w) = mw + \beta$ maps $S_{r,R}$ to $S_{r',R'}$, then

$$\Re F(w) = m\,\Re w + \Re\beta.$$

As $\Re w$ ranges over $(\log r, \log R)$ (width W), the image interval has width $|m|W$. But the target strip has width $W' = \log(R'/r')$. Therefore

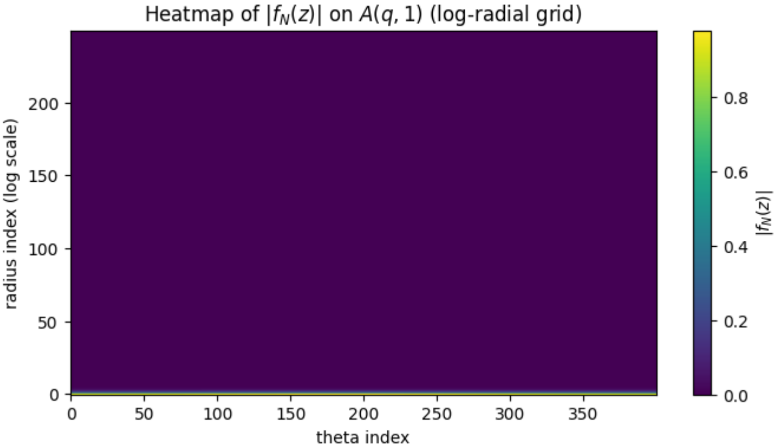
$$W' = |m|W = nW.$$

Divide by 2π:

$$\operatorname{mod}(A') = \frac{W'}{2\pi} = n\,\frac{W}{2\pi} = n \operatorname{mod}(A).$$

Outline

Experiment A



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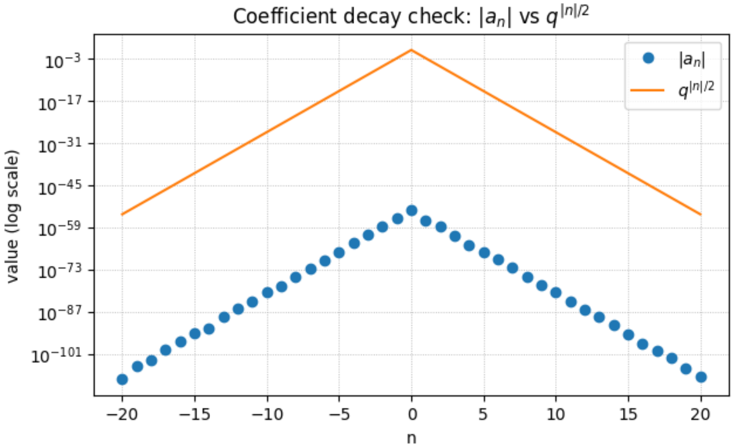

Outline

Conclusion

Thank you!

Questions?

ilt2109@columbia.edu