Chapter 5: Fourier Transform on $\mathbb{R}$
Introduction & Motivation

(Iz Tecúm)

(Columbia University)

December 27, 2025

High-level outline for today

- What problem does the Fourier transform solve?

High-level outline for today

- What problem does the Fourier transform solve?
- Where does its kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ come from?

High-level outline for today

- What problem does the Fourier transform solve?
- Where does its kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ come from?
- From Fourier series (discrete frequencies) to Fourier transform (continuous spectrum)

High-level outline for today

- What problem does the Fourier transform solve?
- Where does its kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ come from?
- From Fourier series (discrete frequencies) to Fourier transform (continuous spectrum)
- Why inversion matters; why it's nontrivial

High-level outline for today

- What problem does the Fourier transform solve?
- Where does its kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ come from?
- From Fourier series (discrete frequencies) to Fourier transform (continuous spectrum)
- Why inversion matters; why it's nontrivial
- Core properties you'll use repeatedly

High-level outline for today

- What problem does the Fourier transform solve?
- Where does its kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ come from?
- From Fourier series (discrete frequencies) to Fourier transform (continuous spectrum)
- Why inversion matters; why it's nontrivial
- Core properties you'll use repeatedly
- Where this chapter sits: PDE, probability, signal processing

Roadmap

What problem are we solving?

Goal: Decompose a function $f : \mathbb{R} \to \mathbb{C}$ into *pure oscillations* (frequencies).

Two worlds

- **Time/space domain** (original x-variable): $f(x)$

What problem are we solving?

Goal: Decompose a function $f : \mathbb{R} \to \mathbb{C}$ into *pure oscillations* (frequencies).

Two worlds

- **Time/space domain** (original x-variable): $f(x)$
- **Frequency domain** (frequency ξ): $\widehat{f}(\xi)$

What problem are we solving?

Goal: Decompose a function $f : \mathbb{R} \to \mathbb{C}$ into *pure oscillations* (frequencies).

Two worlds

- **Time/space domain** (original x-variable): $f(x)$
- **Frequency domain** (frequency ξ): $\widehat{f}(\xi)$

Why switch domains?

- Linear operations that are *hard* in time (e.g. convolution) become *easy* in frequency (pointwise products).

What problem are we solving?

Goal: Decompose a function $f : \mathbb{R} \to \mathbb{C}$ into *pure oscillations* (frequencies).

Two worlds

- **Time/space domain** (original x-variable): $f(x)$
- **Frequency domain** (frequency ξ): $\widehat{f}(\xi)$

Why switch domains?

- Linear operations that are *hard* in time (e.g. convolution) become *easy* in frequency (pointwise products).
- Many PDEs and filters are *diagonal* in frequency.

What problem are we solving?

Goal: Decompose a function $f : \mathbb{R} \to \mathbb{C}$ into *pure oscillations* (frequencies).

Two worlds

- **Time/space domain** (original x-variable): $f(x)$
- **Frequency domain** (frequency ξ): $\widehat{f}(\xi)$

Why switch domains?

- Linear operations that are *hard* in time (e.g. convolution) become *easy* in frequency (pointwise products).
- Many PDEs and filters are *diagonal* in frequency.
- Random variables: distribution $\leftrightarrow$ characteristic function (Fourier transform).

Where does the kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ come from?

Translation symmetry on $\mathbb{R}$: $(T_t f)(x) = f(x-t)$.

Where does the kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ come from?

Translation symmetry on $\mathbb{R}$: $(T_t f)(x) = f(x - t)$.

Eigenfunction requirement

Seek functions ϕ with $T_t\phi = \lambda(t)\phi$ for all $t \in \mathbb{R}$:

$$\phi(x - t) = \lambda(t)\,\phi(x) \quad \forall x, t.$$

Where does the kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ come from?

Translation symmetry on $\mathbb{R}$: $(T_t f)(x) = f(x-t)$.

Eigenfunction requirement

Seek functions ϕ with $T_t\phi = \lambda(t)\phi$ for all $t \in \mathbb{R}$:

$$\phi(x-t) = \lambda(t)\,\phi(x) \quad \forall x, t.$$

Then $\lambda(t+s) = \lambda(t)\lambda(s)$, and with mild regularity $\lambda(t) = \mathrm{e}^{ct}$.

Where does the kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ come from?

Translation symmetry on $\mathbb{R}$: $(T_t f)(x) = f(x - t)$.

Eigenfunction requirement

Seek functions ϕ with $T_t\phi = \lambda(t)\phi$ for all $t \in \mathbb{R}$:

$$\phi(x - t) = \lambda(t)\,\phi(x) \quad \forall x, t.$$

Then $\lambda(t + s) = \lambda(t)\lambda(s)$, and with mild regularity $\lambda(t) = \mathrm{e}^{ct}$.

- To keep $|\lambda(t)| = 1$ (isometries in L^2): $c = 2\pi\mathrm{i}\,\xi$, $\xi \in \mathbb{R}$.
- $\Rightarrow$ eigenfunctions: $\phi_\xi(x) = \mathrm{e}^{2\pi \mathrm{i}\xi x}$.

Where does the kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ come from?

Translation symmetry on $\mathbb{R}$: $(T_t f)(x) = f(x-t)$.

Eigenfunction requirement

Seek functions ϕ with $T_t\phi = \lambda(t)\phi$ for all $t \in \mathbb{R}$:

$$\phi(x-t) = \lambda(t)\,\phi(x) \quad \forall x, t.$$

Then $\lambda(t+s) = \lambda(t)\lambda(s)$, and with mild regularity $\lambda(t) = \mathrm{e}^{ct}$.

- To keep $|\lambda(t)| = 1$ (isometries in L^2): $c = 2\pi\mathrm{i}\,\xi$, $\xi \in \mathbb{R}$.
- $\Rightarrow$ eigenfunctions: $\phi_\xi(x) = \mathrm{e}^{2\pi\mathrm{i}\xi x}$.

Therefore: project f against these characters to read off "how much of frequency ξ" it contains.

From Fourier series to Fourier transform

- On the circle $\mathbb{T}$: $f(x) \sim \sum_{n\in\mathbb{Z}} a_n \, \mathrm{e}^{2\pi \mathrm{i} n x}, \quad a_n = \int_0^1 f(x) \, \mathrm{e}^{-2\pi \mathrm{i} n x} \, dx.$

From Fourier series to Fourier transform

- On the circle $\mathbb{T}$: $f(x) \sim \sum_{n\in\mathbb{Z}} a_n \, \mathrm{e}^{2\pi \mathrm{i} n x}, \quad a_n = \int_0^1 f(x) \, \mathrm{e}^{-2\pi \mathrm{i} n x} \, dx$.
- On $\mathbb{R}$: frequencies become *continuous*.

From Fourier series to Fourier transform

- On the circle $\mathbb{T}$: $f(x) \sim \sum_{n\in\mathbb{Z}} a_n \,\mathrm{e}^{2\pi \mathrm{i} n x}$, $\quad a_n = \int_0^1 f(x)\,\mathrm{e}^{-2\pi \mathrm{i} n x}\, dx$.
- On $\mathbb{R}$: frequencies become *continuous*.

Heuristic: let the period $L \to \infty$. Frequency grid n/L becomes continuous ξ, sums $\to$ integrals.

From Fourier series to Fourier transform

- On the circle $\mathbb{T}$: $f(x) \sim \sum_{n\in\mathbb{Z}} a_n \, \mathrm{e}^{2\pi \mathrm{i} n x}$, $\quad a_n = \int_0^1 f(x)\, \mathrm{e}^{-2\pi \mathrm{i} n x}\, dx$.
- On $\mathbb{R}$: frequencies become *continuous*.

Heuristic: let the period $L \to \infty$. Frequency grid n/L becomes continuous ξ, sums $\to$ integrals.

Definition (Fourier transform on $\mathbb{R}$)

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, dx \quad \text{(at least for } f \in L^1(\mathbb{R})\text{)}.$$

- On the circle $\mathbb{T}$: $f(x) \sim \sum_{n\in\mathbb{Z}} a_n \, \mathrm{e}^{2\pi \mathrm{i} nx}, \quad a_n = \int_0^1 f(x)\,\mathrm{e}^{-2\pi \mathrm{i} nx}\,dx$.
- On $\mathbb{R}$: frequencies become *continuous*.

Heuristic: let the period $L \to \infty$. Frequency grid n/L becomes continuous ξ, sums $\to$ integrals.

Definition (Fourier transform on $\mathbb{R}$)

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-2\pi \mathrm{i} x\xi}\,dx \quad \text{(at least for } f \in L^1(\mathbb{R})\text{)}.$$

Different texts use different 2π conventions; here it's the Stein–Shakarchi normalization.

First properties (why this change of variables is useful)

For nice f, g (e.g. L^1 or Schwartz):

$$\text{Linearity: } (\widehat{af + bg}) = a\,\widehat{f} + b\,\widehat{g}.$$

$$\text{Translation: } \widehat{(T_t f)}(\xi) = \mathrm{e}^{-2\pi \mathrm{i} t\xi}\,\widehat{f}(\xi).$$

$$\text{Scaling: } (\widehat{f(ax)})(\xi) = \frac{1}{|a|}\,\widehat{f}\left(\frac{\xi}{a}\right).$$

$$\text{Convolution: } (\widehat{f * g}) = \widehat{f} \cdot \widehat{g}.$$

$$\text{Differentiation: } \widehat{(f')}(\xi) = (2\pi \mathrm{i}\xi)\,\widehat{f}(\xi).$$

$$\text{Multiplication by } x : (\widehat{xf(x)})(\xi) = \frac{1}{2\pi \mathrm{i}}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{f}(\xi).$$

First properties (why this change of variables is useful)

For nice f, g (e.g. L^1 or Schwartz):

$$\text{Linearity: } (\widehat{af+bg}) = a\,\widehat{f} + b\,\widehat{g}.$$

$$\text{Translation: } \widehat{(T_t f)}(\xi) = \mathrm{e}^{-2\pi \mathrm{i} t\xi}\,\widehat{f}(\xi).$$

$$\text{Scaling: } (\widehat{f(ax)})(\xi) = \frac{1}{|a|}\,\widehat{f}\left(\frac{\xi}{a}\right).$$

$$\text{Convolution: } (\widehat{f*g}) = \widehat{f}\cdot\widehat{g}.$$

$$\text{Differentiation: } \widehat{(f')}(\xi) = (2\pi\mathrm{i}\xi)\,\widehat{f}(\xi).$$

$$\text{Multiplication by } x: (\widehat{xf(x)})(\xi) = \frac{1}{2\pi\mathrm{i}}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{f}(\xi).$$

Meta: convolution $\leftrightarrow$ multiplication; differentiation $\leftrightarrow$ polynomial weights in ξ.

Why inversion matters (and why it's delicate)

- We want $\mathcal{F}^{-1}(\mathcal{F}f) = f$: frequency analysis must be *reversible*.

Why inversion matters (and why it's delicate)

- We want $\mathcal{F}^{-1}(\mathcal{F}f) = f$: frequency analysis must be *reversible*.
- Inversion theorem (informal): if $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi \quad \text{for a.e. } x.$$

Why inversion matters (and why it's delicate)

- We want $\mathcal{F}^{-1}(\mathcal{F}f) = f$: frequency analysis must be *reversible*.
- Inversion theorem (informal): if $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi \quad \text{for a.e. } x.$$

- Delicate because absolute convergence and pointwise limits can fail without hypotheses.

Why inversion matters (and why it's delicate)

- We want $\mathcal{F}^{-1}(\mathcal{F}f) = f$: frequency analysis must be *reversible*.
- Inversion theorem (informal): if $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi \quad \text{for a.e. } x.$$

- Delicate because absolute convergence and pointwise limits can fail without hypotheses.
- Strategy in the chapter: *prove inversion for very nice f* (Schwartz), then *extend by density*.

Two running examples (intuition builders)

Gaussian: $f(x) = \mathrm{e}^{-\pi x^2} \Rightarrow \widehat{f}(\xi) = \mathrm{e}^{-\pi \xi^2}$ (self-dual).

Two running examples (intuition builders)

Gaussian: $f(x) = \mathrm{e}^{-\pi x^2} \Rightarrow \widehat{f}(\xi) = \mathrm{e}^{-\pi \xi^2}$ (self-dual).

Box/rect function: $f = \mathbf{1}_{[-1,1]} \Rightarrow$

$$\widehat{f}(\xi) = \int_{-1}^{1} \mathrm{e}^{-2\pi \mathrm{i} x \xi}\, dx = \frac{\sin(2\pi\xi)}{\pi\xi} \quad \text{(sinc)}.$$

Two running examples (intuition builders)

Gaussian: $f(x) = \mathrm{e}^{-\pi x^2} \Rightarrow \widehat{f}(\xi) = \mathrm{e}^{-\pi \xi^2}$ (self-dual).

Box/rect function: $f = \mathbf{1}_{[-1,1]} \Rightarrow$

$$\widehat{f}(\xi) = \int_{-1}^{1} \mathrm{e}^{-2\pi \mathrm{i} x \xi}\, dx = \frac{\sin(2\pi\xi)}{\pi\xi} \quad \text{(sinc).}$$

Lessons:

- Smooth $\Rightarrow$ fast decay in frequency (Gaussian).
- Sharp edges $\Rightarrow$ slowly decaying oscillations (sinc tails).

Where does this chapter sit (big applications)

- **PDE / Heat equation:** Fourier turns $\partial_t u = \Delta u$ into $\partial_t \widehat{u} = -4\pi^2|\xi|^2\widehat{u}$ (decoupled ODEs).

Where does this chapter sit (big applications)

- **PDE / Heat equation:** Fourier turns $\partial_t u = \Delta u$ into $\partial_t \widehat{u} = -4\pi^2 |\xi|^2 \widehat{u}$ (decoupled ODEs).
- **Probability:** characteristic function $\varphi_X(\xi) = \mathbb{E}[\mathrm{e}^{2\pi \mathrm{i} \xi X}]$ is the Fourier transform of the law of X.

Where does this chapter sit (big applications)

- **PDE / Heat equation:** Fourier turns $\partial_t u = \Delta u$ into $\partial_t \widehat{u} = -4\pi^2|\xi|^2\widehat{u}$ (decoupled ODEs).
- **Probability:** characteristic function $\varphi_X(\xi) = \mathbb{E}[\mathrm{e}^{2\pi\mathrm{i}\xi X}]$ is the Fourier transform of the law of X.
- **Signals/filters:** convolution with a kernel $\Leftrightarrow$ pointwise multiply by a transfer function.

Where does this chapter sit (big applications)

- **PDE / Heat equation:** Fourier turns $\partial_t u = \Delta u$ into $\partial_t \widehat{u} = -4\pi^2 |\xi|^2 \widehat{u}$ (decoupled ODEs).
- **Probability:** characteristic function $\varphi_X(\xi) = \mathbb{E}[\mathrm{e}^{2\pi \mathrm{i} \xi X}]$ is the Fourier transform of the law of X.
- **Signals/filters:** convolution with a kernel $\Leftrightarrow$ pointwise multiply by a transfer function.
- **Uncertainty principles:** cannot be localized tightly in both x and ξ (Heisenberg-type bounds).

What this chapter will / won't do

Will do

- Define $\mathcal{F}$ on $\mathbb{R}$ with Stein–Shakarchi normalization.
- Prove inversion under reasonable L^1 assumptions using approximation.
- Prepare ground for L^2: Plancherel/Parseval in the next chapter.

What this chapter will / won't do

Will do

- Define $\mathcal{F}$ on $\mathbb{R}$ with Stein–Shakarchi normalization.
- Prove inversion under reasonable L^1 assumptions using approximation.
- Prepare ground for L^2: Plancherel/Parseval in the next chapter.

Won't (yet)

- Full distribution theory and tempered distributions (later courses).
- Multidimensional transforms in generality (arrive later).
- Deep singular integral theory (needs more machinery).

Learning roadmap for the chapter

1. Master the *definition* + core properties by rote (they drive every computation).

Learning roadmap for the chapter

1. Master the *definition* + core properties by rote (they drive every computation).
2. Understand the *inversion proof strategy*: nice class $\rightarrow$ approximation $\rightarrow$ limit.

Learning roadmap for the chapter

1. Master the *definition* $+$ core properties by rote (they drive every computation).
2. Understand the *inversion proof strategy*: nice class $\rightarrow$ approximation $\rightarrow$ limit.
3. Drill the *two canon examples* (Gaussian, box $\rightarrow$ sinc).

Learning roadmap for the chapter

1. Master the *definition* $+$ core properties by rote (they drive every computation).
2. Understand the *inversion proof strategy*: nice class $\to$ approximation $\to$ limit.
3. Drill the *two canon examples* (Gaussian, box $\to$ sinc).
4. Keep applications in mind: PDE (heat), probability (characteristic functions), filtering.

Micro summary (what to remember from this intro)

- Kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ is forced by translation symmetry (characters of $(\mathbb{R}, +)$).

Micro summary (what to remember from this intro)

- Kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ is forced by translation symmetry (characters of $(\mathbb{R}, +)$).
- $\mathcal{F}$ transforms convolution to multiplication and derivatives to polynomials in ξ.

Micro summary (what to remember from this intro)

- Kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ is forced by translation symmetry (characters of $(\mathbb{R}, +)$).
- $\mathcal{F}$ transforms convolution to multiplication and derivatives to polynomials in ξ.
- Inversion is the central theorem; proofs proceed by approximation from very nice functions.

Micro summary (what to remember from this intro)

- Kernel $\mathrm{e}^{-2\pi \mathrm{i} x\xi}$ is forced by translation symmetry (characters of $(\mathbb{R}, +)$).
- $\mathcal{F}$ transforms convolution to multiplication and derivatives to polynomials in ξ.
- Inversion is the central theorem; proofs proceed by approximation from very nice functions.
- This unlocks PDE, probability, and signal processing viewpoints.

Roadmap

Setup & domain

Fourier transform on $\mathbb{R}$ (Stein–Shakarchi normalization)

For $f \in L^1(\mathbb{R})$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, dx.$$

Setup & domain

Fourier transform on $\mathbb{R}$ (Stein–Shakarchi normalization)

For $f \in L^1(\mathbb{R})$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, dx.$$

- **Minimal domain:** $L^1(\mathbb{R})$ ensures the integral is absolutely convergent.

Setup & domain

Fourier transform on $\mathbb{R}$ (Stein–Shakarchi normalization)

For $f \in L^1(\mathbb{R})$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x \xi}\, dx.$$

- **Minimal domain:** $L^1(\mathbb{R})$ ensures the integral is absolutely convergent.
- **Nice class:** Schwartz $\mathcal{S}(\mathbb{R})$ (smooth, rapidly decaying) is dense in L^p and stable under $\mathcal{F}$.

Setup & domain

Fourier transform on $\mathbb{R}$ (Stein–Shakarchi normalization)

For $f \in L^1(\mathbb{R})$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, dx.$$

- **Minimal domain:** $L^1(\mathbb{R})$ ensures the integral is absolutely convergent.
- **Nice class:** Schwartz $\mathcal{S}(\mathbb{R})$ (smooth, rapidly decaying) is dense in L^p and stable under $\mathcal{F}$.
- **Conventions:** Different books place 2π differently. We stick to Stein–Shakarchi.

Linearity, conjugation, reflection

For $f, g \in L^1$, $a, b \in \mathbb{C}$:

$$(\widehat{af + bg}) = a\,\widehat{f} + b\,\widehat{g}.$$

Linearity, conjugation, reflection

For $f, g \in L^1$, $a, b \in \mathbb{C}$:

$$\widehat{(af + bg)} = a\,\widehat{f} + b\,\widehat{g}.$$

If $\overline{f}$ is the complex conjugate of f,

$$\widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)}.$$

Linearity, conjugation, reflection

For $f, g \in L^1$, $a, b \in \mathbb{C}$:

$$\widehat{(af + bg)} = a\,\widehat{f} + b\,\widehat{g}.$$

If $\overline{f}$ is the complex conjugate of f,

$$\widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)}.$$

If $f^\sharp(x) := f(-x)$ (time-reversal),

$$\widehat{f^\sharp}(\xi) = \widehat{f}(-\xi).$$

Linearity, conjugation, reflection

For $f, g \in L^1$, $a, b \in \mathbb{C}$:

$$\widehat{(af + bg)} = a\,\widehat{f} + b\,\widehat{g}.$$

If $\overline{f}$ is the complex conjugate of f,

$$\widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)}.$$

If $f^\sharp(x) := f(-x)$ (time-reversal),

$$\widehat{f^\sharp}(\xi) = \widehat{f}(-\xi).$$

Real/even/odd patterns:

- f real $\Rightarrow \widehat{f}(-\xi) = \overline{\widehat{f}(\xi)}$ (Hermitian symmetry).
- f even real $\Rightarrow \widehat{f}$ real and even.
- f odd real $\Rightarrow \widehat{f}$ purely imaginary and odd.

Translation & modulation (prove it)

Translation: $(T_t f)(x) := f(x-t)$.

$$\widehat{(T_t f)}(\xi) = \int_{\mathbb{R}} f(x-t)\,\mathrm{e}^{-2\pi \mathrm{i} x\xi} dx \overset{u=x-t}{=} \mathrm{e}^{-2\pi \mathrm{i} t\xi} \int_{\mathbb{R}} f(u)\,\mathrm{e}^{-2\pi \mathrm{i} u\xi} du = \mathrm{e}^{-2\pi \mathrm{i} t\xi}\,\widehat{f}(\xi).$$

Translation & modulation (prove it)

Translation: $(T_t f)(x) := f(x-t)$.

$$\widehat{(T_t f)}(\xi) = \int_{\mathbb{R}} f(x-t)\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,dx \overset{u=x-t}{=} \mathrm{e}^{-2\pi\mathrm{i}t\xi} \int_{\mathbb{R}} f(u)\,\mathrm{e}^{-2\pi\mathrm{i}u\xi}\,du = \mathrm{e}^{-2\pi\mathrm{i}t\xi}\,\widehat{f}(\xi).$$

Modulation: $M_{\xi_0} f(x) := \mathrm{e}^{2\pi\mathrm{i}\xi_0 x} f(x)$.

$$\widehat{(M_{\xi_0} f)}(\xi) = \int f(x)\,\mathrm{e}^{2\pi\mathrm{i}\xi_0 x}\,\mathrm{e}^{-2\pi\mathrm{i}\xi x}\,dx = \widehat{f}(\xi - \xi_0).$$

Moral: shifts in time $\leftrightarrow$ phase in frequency; tones in time $\leftrightarrow$ shifts in frequency.

Scaling & time reversal (derive)

For $a \neq 0$, define $S_a f(x) := f(ax)$.

$$\widehat{(S_a f)}(\xi) = \int f(ax)\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}dx \overset{u=ax}{=} \frac{1}{|a|}\int f(u)\,\mathrm{e}^{-2\pi\mathrm{i}\,u\,(\xi/a)}du = \frac{1}{|a|}\,\widehat{f}\left(\frac{\xi}{a}\right).$$

Scaling & time reversal (derive)

For $a \neq 0$, define $S_a f(x) := f(ax)$.

$$\widehat{(S_a f)}(\xi) = \int f(ax)\, \mathrm{e}^{-2\pi \mathrm{i} x \xi} dx \overset{u=ax}{=} \frac{1}{|a|} \int f(u)\, \mathrm{e}^{-2\pi \mathrm{i}\, u\,(\xi/a)} du = \frac{1}{|a|}\, \widehat{f}\left(\frac{\xi}{a}\right).$$

Time reversal $f^\sharp(x) = f(-x)$ was: $\widehat{f^\sharp}(\xi) = \widehat{f}(-\xi)$. **Heuristic:** zoom in time $\Rightarrow$ stretch out in frequency (and renormalize).

Differentiation & multiplication (operational rules)

If $f, f' \in L^1$,

$$\widehat{(f')}(\xi) = \int f'(x)\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}dx \overset{\text{IBP}}{=} \Big[f(x)\mathrm{e}^{-2\pi\mathrm{i}x\xi}\Big]_{-\infty}^{\infty} + (2\pi\mathrm{i}\xi)\int f(x)\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}dx = (2\pi\mathrm{i}\xi)\widehat{f}(\xi),$$

where the boundary term vanishes for $f \in \mathcal{S}$; for L^1 use approximation.

Differentiation & multiplication (operational rules)

If $f, f' \in L^1$,

$$\widehat{(f')}(\xi) = \int f'(x)\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}dx \overset{\text{IBP}}{=} \Big[f(x)\mathrm{e}^{-2\pi\mathrm{i}x\xi}\Big]_{-\infty}^{\infty} + (2\pi\mathrm{i}\xi)\int f(x)\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}dx = (2\pi\mathrm{i}\xi)\widehat{f}(\xi),$$

where the boundary term vanishes for $f \in \mathcal{S}$; for L^1 use approximation.

$$(\widehat{xf(x)})(\xi) = \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{f}(\xi) \quad \text{(differentiate under the integral).}$$

Meta: derivatives $\leftrightarrow$ polynomial factors in ξ; moments in time $\leftrightarrow$ derivatives in frequency.

Convolution theorem (key bridge)

$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\,dy$ with $f, g \in L^1$ gives $f * g \in L^1$.

$$\widehat{(f * g)}(\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,dy\,dx \overset{x-y=u}{=} \int g(y)\,\mathrm{e}^{-2\pi\mathrm{i}y\xi}dy \cdot \int f(u)\,\mathrm{e}^{-2\pi\mathrm{i}u\xi}du = \widehat{f}(\xi)\,\widehat{g}($$

Convolution theorem (key bridge)

$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\,dy$ with $f, g \in L^1$ gives $f * g \in L^1$.

$$\widehat{(f * g)}(\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{e}^{-2\pi \mathrm{i} x\xi}\,dy\,dx \overset{x-y=u}{=} \int g(y)\,\mathrm{e}^{-2\pi \mathrm{i} y\xi} dy \cdot \int f(u)\,\mathrm{e}^{-2\pi \mathrm{i} u\xi} du = \widehat{f}(\xi)\,\widehat{g}($$

Corollary: $f * g$ is a *smoothed* version of f when g is smooth/decaying; in frequency you just multiply by $\widehat{g}$ (a low-pass filter).

Riemann–Lebesgue lemma (decay)

Lemma

If $f \in L^1(\mathbb{R})$ then $\widehat{f}$ is bounded, uniformly continuous, and

$$\widehat{f}(\xi) \to 0 \quad \text{as } |\xi| \to \infty.$$

Riemann–Lebesgue lemma (decay)

Lemma

If $f \in L^1(\mathbb{R})$ then $\widehat{f}$ is bounded, uniformly continuous, and

$$\widehat{f}(\xi) \to 0 \quad \text{as } |\xi| \to \infty.$$

Sketch:

- Boundedness by $\|\widehat{f}\|_\infty \leq \|f\|_1$ (triangle inequality).
- Approximate f by compactly supported continuous functions; oscillations cancel at large $|\xi|$ (Dirichlet kernel trick).
- Uniform continuity from dominated convergence (difference quotient inside the integral).

Intuition: integrals against faster oscillations average out.

Symmetry & uncertainty (preview)

- **Even/odd & real:** summarized earlier (Hermitian symmetry).

Symmetry & uncertainty (preview)

- **Even/odd & real:** summarized earlier (Hermitian symmetry).
- **Time–frequency tradeoff (Heisenberg):**

$$\sigma_x(f)\,\sigma_\xi(\widehat{f}) \;\geq\; \frac{1}{4\pi},$$

where σ denotes standard deviation (for L^2-normalized f).

Symmetry & uncertainty (preview)

- **Even/odd & real:** summarized earlier (Hermitian symmetry).
- **Time–frequency tradeoff (Heisenberg):**

$$\sigma_x(f)\,\sigma_\xi(\widehat{f}) \;\geq\; \frac{1}{4\pi},$$

  where σ denotes standard deviation (for L^2-normalized f).
- Equality for Gaussians $\Rightarrow$ optimally concentrated in both domains.

Worked example I: Gaussian is self-dual

Let $f(x) = \mathrm{e}^{-\pi x^2}$. Complete the square:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\pi x^2}\, \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, dx = \int_{\mathbb{R}} \mathrm{e}^{-\pi\,(x^2+2\mathrm{i}x\xi)}\, dx = \int_{\mathbb{R}} \mathrm{e}^{-\pi\,(x-\mathrm{i}\xi)^2}\, \mathrm{e}^{-\pi\xi^2}\, dx.$$

Worked example I: Gaussian is self-dual

Let $f(x) = \mathrm{e}^{-\pi x^2}$. Complete the square:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\pi x^2}\, \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, dx = \int_{\mathbb{R}} \mathrm{e}^{-\pi\,(x^2+2\mathrm{i}x\xi)}\, dx = \int_{\mathbb{R}} \mathrm{e}^{-\pi\,(x-\mathrm{i}\xi)^2}\, \mathrm{e}^{-\pi\xi^2}\, dx.$$

Shift the contour (Schwarz class justification) or use standard Gaussian integral:

$$\int_{\mathbb{R}} \mathrm{e}^{-\pi\,(x-\mathrm{i}\xi)^2}\, dx = \int_{\mathbb{R}} \mathrm{e}^{-\pi u^2}\, du = 1.$$

$$\Rightarrow \quad \widehat{f}(\xi) = \mathrm{e}^{-\pi\xi^2}.$$

Takeaway: Gaussians minimize uncertainty and are fixed by $\mathcal{F}$.

Worked example II: Box $\to$ sinc

$f(x) = \mathbf{1}_{[-1,1]}(x)$.

$$\widehat{f}(\xi) = \int_{-1}^{1} \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, dx = \left[\frac{\mathrm{e}^{-2\pi \mathrm{i} x\xi}}{-2\pi \mathrm{i}\xi}\right]_{-1}^{1} = \frac{\mathrm{e}^{-2\pi \mathrm{i}\xi} - \mathrm{e}^{2\pi \mathrm{i}\xi}}{-2\pi \mathrm{i}\xi} = \frac{\sin(2\pi\xi)}{\pi\xi}.$$

Worked example II: Box $\to$ sinc

$f(x) = \mathbf{1}_{[-1,1]}(x)$.

$$\widehat{f}(\xi) = \int_{-1}^{1} \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, dx = \left[\frac{\mathrm{e}^{-2\pi \mathrm{i} x\xi}}{-2\pi \mathrm{i}\xi}\right]_{-1}^{1} = \frac{\mathrm{e}^{-2\pi \mathrm{i}\xi} - \mathrm{e}^{2\pi \mathrm{i}\xi}}{-2\pi \mathrm{i}\xi} = \frac{\sin(2\pi\xi)}{\pi\xi}.$$

Observations:

- Decays like $1/|\xi|$ (slow, due to discontinuities).
- Zeros at $\xi \in \mathbb{Z} \setminus \{0\}$.
- Sinc tails $\Rightarrow$ Gibbs-type ripples on reconstruction near jumps.

Worked example III: Triangle via convolution

Triangle $\tau = \mathbf{1}_{[-1,1]} * \mathbf{1}_{[-1,1]}$.

$$\widehat{\tau}(\xi) = \widehat{\mathbf{1}_{[-1,1]}}(\xi) \cdot \widehat{\mathbf{1}_{[-1,1]}}(\xi) = \left(\frac{\sin(2\pi\xi)}{\pi\xi} \right)^2 .$$

Worked example III: Triangle via convolution

Triangle $\tau = \mathbf{1}_{[-1,1]} * \mathbf{1}_{[-1,1]}$.

$$\widehat{\tau}(\xi) = \widehat{\mathbf{1}_{[-1,1]}}(\xi) \cdot \widehat{\mathbf{1}_{[-1,1]}}(\xi) = \left(\frac{\sin(2\pi\xi)}{\pi\xi}\right)^2 .$$

Moral: Convolution in time smooths the edges $\Rightarrow$ faster decay (here $\sim 1/\xi^2$) in frequency.

Common pitfalls & domain cautions

- Not every L^1 function has $\widehat{f} \in L^1$; inversion needs care (will prove under extra hypotheses).

Common pitfalls & domain cautions

- Not every L^1 function has $\widehat{f} \in L^1$; inversion needs care (will prove under extra hypotheses).
- Functions with jumps decay slowly in frequency ($1/\xi$ tails).

Common pitfalls & domain cautions

- Not every L^1 function has $\widehat{f} \in L^1$; inversion needs care (will prove under extra hypotheses).
- Functions with jumps decay slowly in frequency ($1/\xi$ tails).
- Poles/impulses (e.g. δ) live in the world of distributions; we postpone this framework.

Common pitfalls & domain cautions

- Not every L^1 function has $\widehat{f} \in L^1$; inversion needs care (will prove under extra hypotheses).
- Functions with jumps decay slowly in frequency ($1/\xi$ tails).
- Poles/impulses (e.g. δ) live in the world of distributions; we postpone this framework.
- Interchange of integrals (Fubini/Tonelli) requires integrability checks; we always justify via L^1 or Schwartz approximation.

Bridge to the next section

- We now have the toolkit: translation/modulation, scaling, convolution, differentiation rules.

Bridge to the next section

- We now have the toolkit: translation/modulation, scaling, convolution, differentiation rules.
- Next: **Inversion theorem** — when and how $\mathcal{F}^{-1}(\mathcal{F}f) = f$.

Bridge to the next section

- We now have the toolkit: translation/modulation, scaling, convolution, differentiation rules.
- Next: **Inversion theorem** — when and how $\mathcal{F}^{-1}(\mathcal{F}f) = f$.
- Strategy preview: prove for Schwartz, then extend to L^1 by approximation (Gaussian mollifiers / dominated convergence).

Roadmap

The inversion problem

Question

Given $f : \mathbb{R} \to \mathbb{C}$, can we *recover* f from its Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x \xi}\, dx\,?$$

The inversion problem

Question

Given $f : \mathbb{R} \to \mathbb{C}$, can we *recover* f from its Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x \xi}\, dx\,?$$

- Formal guess: $f(x) \stackrel{?}{=} \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x \xi}\, d\xi.$

The inversion problem

Question

Given $f : \mathbb{R} \to \mathbb{C}$, can we *recover* f from its Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} x\xi}\, dx\,?$$

- Formal guess: $f(x) \stackrel{?}{=} \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi.$
- Issues: absolute convergence; interchanging integrals; pointwise vs. a.e. equality.

The inversion problem

Question

Given $f : \mathbb{R} \to \mathbb{C}$, can we *recover* f from its Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\,\mathrm{e}^{-2\pi \mathrm{i} x\xi}\,dx\,?$$

- Formal guess: $f(x) \stackrel{?}{=} \int_{\mathbb{R}} \widehat{f}(\xi)\,\mathrm{e}^{2\pi \mathrm{i} x\xi}\,d\xi$.
- Issues: absolute convergence; interchanging integrals; pointwise vs. a.e. equality.
- Strategy (Chapter 5): prove for *very nice f* (Schwartz), extend by approximation.

A standard sufficient condition

Fourier inversion (one L^1 version)

If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x \xi}\, d\xi \quad \text{for a.e. } x \in \mathbb{R}.$$

A standard sufficient condition

Fourier inversion (one L^1 version)

If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi \quad \text{for a.e. } x \in \mathbb{R}.$$

Notes:

- The a.e. qualifier is sharp in general.
- There are many variants (e.g. pointwise at continuity points).
- The L^2-theory (Plancherel) gives a clean inverse on L^2—next chapter.

Approximate identities via Gaussians

Base Gaussian

$g(x) := \mathrm{e}^{-\pi x^2}$ satisfies $\widehat{g}(\xi) = g(\xi)$.

Approximate identities via Gaussians

Base Gaussian

$g(x) := \mathrm{e}^{-\pi x^2}$ satisfies $\widehat{g}(\xi) = g(\xi)$.

Scaled approximate identity

Define $\phi_\varepsilon(x) := \frac{1}{\varepsilon}\, g\left(\frac{x}{\varepsilon}\right) = \frac{1}{\varepsilon}\, \mathrm{e}^{-\pi(x/\varepsilon)^2}$, so $\int_{\mathbb{R}} \phi_\varepsilon = 1$ and $\phi_\varepsilon \to \delta$ (weakly).

Approximate identities via Gaussians

Base Gaussian

$g(x) := \mathrm{e}^{-\pi x^2}$ satisfies $\widehat{g}(\xi) = g(\xi)$.

Scaled approximate identity

Define $\phi_\varepsilon(x) := \dfrac{1}{\varepsilon}\, g\left(\dfrac{x}{\varepsilon}\right) = \dfrac{1}{\varepsilon}\,\mathrm{e}^{-\pi(x/\varepsilon)^2}$, so $\displaystyle\int_{\mathbb{R}} \phi_\varepsilon = 1$ and $\phi_\varepsilon \to \delta$ (weakly).

- **Fourier side:** by scaling,
$$\widehat{\phi_\varepsilon}(\xi) = \widehat{g}(\varepsilon\xi) = \mathrm{e}^{-\pi\varepsilon^2\xi^2}.$$
- **Consequence:** $f * \phi_\varepsilon \to f$ in L^1 and pointwise at Lebesgue points.

Key identity linking $f * \phi_\varepsilon$ and $\widehat{f}$

For $f \in L^1(\mathbb{R})$ and $\varepsilon > 0$:

$$(f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x \xi}\, \widehat{\phi_\varepsilon}(\xi)\, d\xi = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x \xi}\, \mathrm{e}^{-\pi \varepsilon^2 \xi^2}\, d\xi.$$

Key identity linking $f * \phi_\varepsilon$ and $\widehat{f}$

For $f \in L^1(\mathbb{R})$ and $\varepsilon > 0$:

$$(f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \widehat{\phi_\varepsilon}(\xi)\, d\xi = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi.$$

Why true? For nice f (e.g. Schwartz), $\widehat{f * \phi_\varepsilon} = \widehat{f} \cdot \widehat{\phi_\varepsilon}$; take inverse transform. For $f \in L^1$, use approximation by Schwartz and dominated convergence (details next).

Proof skeleton: Schwartz case first

Let $f \in \mathcal{S}(\mathbb{R})$.

1. $\mathcal{S}$ is invariant under $\mathcal{F}$ and convolution; $\widehat{g} = g$ for $g(x) = \mathrm{e}^{-\pi x^2}$.

Proof skeleton: Schwartz case first

Let $f \in \mathcal{S}(\mathbb{R})$.

1. $\mathcal{S}$ is invariant under $\mathcal{F}$ and convolution; $\widehat{g} = g$ for $g(x) = \mathrm{e}^{-\pi x^2}$.
2. For $\phi_\varepsilon(x) = \varepsilon^{-1} g(x/\varepsilon)$ we have $\widehat{\phi_\varepsilon}(\xi) = \mathrm{e}^{-\pi \varepsilon^2 \xi^2}$.

Proof skeleton: Schwartz case first

Let $f \in \mathcal{S}(\mathbb{R})$.

1. $\mathcal{S}$ is invariant under $\mathcal{F}$ and convolution; $\widehat{g} = g$ for $g(x) = \mathrm{e}^{-\pi x^2}$.
2. For $\phi_\varepsilon(x) = \varepsilon^{-1} g(x/\varepsilon)$ we have $\widehat{\phi_\varepsilon}(\xi) = \mathrm{e}^{-\pi\varepsilon^2\xi^2}$.
3. Convolution theorem: $\widehat{f * \phi_\varepsilon} = \widehat{f} \cdot \widehat{\phi_\varepsilon}$.

Proof skeleton: Schwartz case first

Let $f \in \mathcal{S}(\mathbb{R})$.

1. $\mathcal{S}$ is invariant under $\mathcal{F}$ and convolution; $\widehat{g} = g$ for $g(x) = \mathrm{e}^{-\pi x^2}$.
2. For $\phi_\varepsilon(x) = \varepsilon^{-1} g(x/\varepsilon)$ we have $\widehat{\phi_\varepsilon}(\xi) = \mathrm{e}^{-\pi\varepsilon^2\xi^2}$.
3. Convolution theorem: $\widehat{f * \phi_\varepsilon} = \widehat{f} \cdot \widehat{\phi_\varepsilon}$.
4. Invert (still in $\mathcal{S}$):
$$(f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi.$$

Proof skeleton: Schwartz case first

Let $f \in \mathcal{S}(\mathbb{R})$.

1. $\mathcal{S}$ is invariant under $\mathcal{F}$ and convolution; $\widehat{g} = g$ for $g(x) = \mathrm{e}^{-\pi x^2}$.
2. For $\phi_\varepsilon(x) = \varepsilon^{-1} g(x/\varepsilon)$ we have $\widehat{\phi_\varepsilon}(\xi) = \mathrm{e}^{-\pi\varepsilon^2\xi^2}$.
3. Convolution theorem: $\widehat{f * \phi_\varepsilon} = \widehat{f} \cdot \widehat{\phi_\varepsilon}$.
4. Invert (still in $\mathcal{S}$):
$$(f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi\mathrm{i}x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi.$$
5. As $\varepsilon \to 0$, $\mathrm{e}^{-\pi\varepsilon^2\xi^2} \to 1$ pointwise and is ≤ 1—use dominated convergence (since $\widehat{f} \in \mathcal{S} \subset L^1$) to get
$$\lim_{\varepsilon \to 0} (f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi\mathrm{i}x\xi}\, d\xi.$$

Proof skeleton: Schwartz case first

Let $f \in \mathcal{S}(\mathbb{R})$.

1. $\mathcal{S}$ is invariant under $\mathcal{F}$ and convolution; $\widehat{g} = g$ for $g(x) = \mathrm{e}^{-\pi x^2}$.
2. For $\phi_\varepsilon(x) = \varepsilon^{-1} g(x/\varepsilon)$ we have $\widehat{\phi_\varepsilon}(\xi) = \mathrm{e}^{-\pi\varepsilon^2\xi^2}$.
3. Convolution theorem: $\widehat{f * \phi_\varepsilon} = \widehat{f} \cdot \widehat{\phi_\varepsilon}$.
4. Invert (still in $\mathcal{S}$):
$$(f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi\mathrm{i}x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi.$$
5. As $\varepsilon \to 0$, $\mathrm{e}^{-\pi\varepsilon^2\xi^2} \to 1$ pointwise and is ≤ 1—use dominated convergence (since $\widehat{f} \in \mathcal{S} \subset L^1$) to get
$$\lim_{\varepsilon \to 0} (f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi\mathrm{i}x\xi}\, d\xi.$$
6. But $f * \phi_\varepsilon \to f$ uniformly (even in $\mathcal{S}$-topology), hence inversion holds for Schwartz f.

Extension to L^1 with $\widehat{f} \in L^1$

Let $f \in L^1(\mathbb{R})$ with $\widehat{f} \in L^1(\mathbb{R})$. For each $\varepsilon > 0$:

$$(f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi.$$

Extension to L^1 with $\widehat{f} \in L^1$

Let $f \in L^1(\mathbb{R})$ with $\widehat{f} \in L^1(\mathbb{R})$. For each $\varepsilon > 0$:

$$(f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi.$$

Right-hand side: dominated by $|\widehat{f}(\xi)|$, integrable by hypothesis; thus

$$\lim_{\varepsilon\to 0} \int \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi = \int \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi \quad \text{(DCT)}.$$

Extension to L^1 with $\widehat{f} \in L^1$

Let $f \in L^1(\mathbb{R})$ with $\widehat{f} \in L^1(\mathbb{R})$. For each $\varepsilon > 0$:

$$(f * \phi_\varepsilon)(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi.$$

Right-hand side: dominated by $|\widehat{f}(\xi)|$, integrable by hypothesis; thus

$$\lim_{\varepsilon\to 0} \int \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi = \int \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi \quad \text{(DCT)}.$$

Left-hand side: $f * \phi_\varepsilon \to f$ in L^1 and at *Lebesgue points* of f. Hence for a.e. x,

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, d\xi.$$

Lebesgue points & pointwise recovery

Lebesgue point

x is a Lebesgue point of $f \in L^1$ if

$$\lim_{r \downarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y) - f(x)| \, dy = 0.$$

Lebesgue points & pointwise recovery

Lebesgue point

x is a Lebesgue point of $f \in L^1$ if

$$\lim_{r \downarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y) - f(x)|\, dy = 0.$$

- Almost every x is a Lebesgue point (Lebesgue differentiation theorem).
- For approximate identities (ϕ_ε) with $\int \phi_\varepsilon = 1$, we have $(f * \phi_\varepsilon)(x) \to f(x)$ at Lebesgue points.
- This is the engine behind the a.e. inversion conclusion.

Technical justifications (what we used)

- **Tonelli/Fubini:** We swap integrals in convolution and transform definitions only when the integrand is absolutely integrable; else approximate via $\mathcal{S}$.

Technical justifications (what we used)

- **Tonelli/Fubini:** We swap integrals in convolution and transform definitions only when the integrand is absolutely integrable; else approximate via $\mathcal{S}$.
- **Dominated convergence (DCT):** $\left|\widehat{f}(\xi)\,\mathrm{e}^{-\pi\varepsilon^2\xi^2}\right| \le |\widehat{f}(\xi)| \in L^1$.

Technical justifications (what we used)

- **Tonelli/Fubini:** We swap integrals in convolution and transform definitions only when the integrand is absolutely integrable; else approximate via $\mathcal{S}$.
- **Dominated convergence (DCT):** $\left|\widehat{f}(\xi)\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\right| \le |\widehat{f}(\xi)| \in L^1$.
- **Approximate identity:** ϕ_ε is positive, integrates to 1, and concentrates at 0; hence $f * \phi_\varepsilon \to f$ in L^1 and a.e.

Technical justifications (what we used)

- **Tonelli/Fubini:** We swap integrals in convolution and transform definitions only when the integrand is absolutely integrable; else approximate via $\mathcal{S}$.
- **Dominated convergence (DCT):** $\left|\widehat{f}(\xi)\,\mathrm{e}^{-\pi\varepsilon^2\xi^2}\right| \le |\widehat{f}(\xi)| \in L^1$.
- **Approximate identity:** ϕ_ε is positive, integrates to 1, and concentrates at 0; hence $f * \phi_\varepsilon \to f$ in L^1 and a.e.
- **Schwartz density:** $\mathcal{S}(\mathbb{R})$ is dense in L^1 and stable under $\mathcal{F}$.

Variants & remarks

- If $f \in L^1$ is continuous at x, the inversion formula often holds *pointwise at* x under milder hypotheses.

Variants & remarks

- If $f \in L^1$ is continuous at x, the inversion formula often holds *pointwise at* x under milder hypotheses.
- If $f, \widehat{f} \in L^1$, we get inversion a.e. and *uniform* convergence of $(f * \phi_\varepsilon)$ to f on continuity sets.

Variants & remarks

- If $f \in L^1$ is continuous at x, the inversion formula often holds *pointwise at* x under milder hypotheses.
- If $f, \widehat{f} \in L^1$, we get inversion a.e. and *uniform* convergence of $(f * \phi_\varepsilon)$ to f on continuity sets.
- L^2-inversion (Plancherel) gives a unitary isomorphism on L^2 without needing $\widehat{f} \in L^1$—next chapter.

Variants & remarks

- If $f \in L^1$ is continuous at x, the inversion formula often holds *pointwise at* x under milder hypotheses.
- If $f, \widehat{f} \in L^1$, we get inversion a.e. and *uniform* convergence of $(f * \phi_\varepsilon)$ to f on continuity sets.
- L^2-inversion (Plancherel) gives a unitary isomorphism on L^2 without needing $\widehat{f} \in L^1$—next chapter.
- Edge behavior: discontinuities create oscillatory reconstruction (Gibbs-type phenomena).

Worked example: inversion with Gaussian damping

Suppose $f \in L^1$ and $\widehat{f} \in L^1$.

$$f(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x \xi} \underbrace{\mathrm{e}^{-\pi \varepsilon^2 \xi^2}}_{\text{damps high freq}} \, d\xi.$$

Worked example: inversion with Gaussian damping

Suppose $f \in L^1$ and $\widehat{f} \in L^1$.

$$f(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x \xi} \underbrace{\mathrm{e}^{-\pi \varepsilon^2 \xi^2}}_{\text{damps high freq}} d\xi.$$

Why useful? Numerically stable: the Gaussian factor regularizes the integral; then send $\varepsilon \downarrow 0$.

- If f is smooth/decaying, the damping is harmless and accelerates convergence.
- If f has jumps, damping controls oscillations in the reconstruction.

Quick check: box $\to$ sinc and back (heuristic)

For $f = \mathbf{1}_{[-1,1]}$, $\widehat{f}(\xi) = \dfrac{\sin(2\pi\xi)}{\pi\xi}$.

$$f(x) \approx \int_{\mathbb{R}} \frac{\sin(2\pi\xi)}{\pi\xi}\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi \ \xrightarrow[\varepsilon\downarrow 0]{} \ f(x)$$

at all continuity points $x \neq \pm 1$. At $x = \pm 1$ one gets midpoint limits.

Quick check: box $\to$ sinc and back (heuristic)

For $f = \mathbf{1}_{[-1,1]}$, $\widehat{f}(\xi) = \dfrac{\sin(2\pi\xi)}{\pi\xi}$.

$$f(x) \approx \int_{\mathbb{R}} \frac{\sin(2\pi\xi)}{\pi\xi}\, \mathrm{e}^{2\pi \mathrm{i} x\xi}\, \mathrm{e}^{-\pi\varepsilon^2\xi^2}\, d\xi \ \xrightarrow[\varepsilon\downarrow 0]{}\ f(x)$$

at all continuity points $x \neq \pm 1$. At $x = \pm 1$ one gets midpoint limits. **Moral:** damping $+\ \varepsilon \downarrow 0$ realizes inversion even for nonsmooth f.

Summary: inversion section

- Proved inversion for $\mathcal{S}$ via Gaussian approximate identity.

Summary: inversion section

- Proved inversion for $\mathcal{S}$ via Gaussian approximate identity.
- Extended to $f \in L^1$ with $\widehat{f} \in L^1$ using DCT + Lebesgue points.

Summary: inversion section

- Proved inversion for $\mathcal{S}$ via Gaussian approximate identity.
- Extended to $f \in L^1$ with $\widehat{f} \in L^1$ using DCT + Lebesgue points.
- Learned a practical reconstruction formula with Gaussian damping.

Summary: inversion section

- Proved inversion for $\mathcal{S}$ via Gaussian approximate identity.
- Extended to $f \in L^1$ with $\widehat{f} \in L^1$ using DCT $+$ Lebesgue points.
- Learned a practical reconstruction formula with Gaussian damping.
- Next: L^2 theory (Plancherel), Hausdorff–Young, and applications.

The Fourier Inversion Theorem

Statement

If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then for almost every $x \in \mathbb{R}$,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{ix\xi}\, d\xi.$$

The Fourier Inversion Theorem

Statement

If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then for almost every $x \in \mathbb{R}$,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{ix\xi} \, d\xi.$$

- The Fourier transform is **invertible**.
- Frequency and spatial domains are two sides of the same coin.
- This closes the circle: $f \mapsto \widehat{f} \mapsto f$.

Significance of Fourier Inversion

- **Theoretical:** Proves the transform is not just a computational trick, but a true dual representation.
- **Analysis:** Central in PDEs, signal processing, harmonic analysis.
- **Applications:** Image compression, MRI reconstruction, quantum mechanics, option pricing.

Significance of Fourier Inversion

- **Theoretical:** Proves the transform is not just a computational trick, but a true dual representation.
- **Analysis:** Central in PDEs, signal processing, harmonic analysis.
- **Applications:** Image compression, MRI reconstruction, quantum mechanics, option pricing.

Key Idea

The inversion formula shows that Fourier analysis is a *complete language* for functions.

Further Directions (Not Covered Today)

- **Plancherel Theorem:** Extension of inversion to $L^2(\mathbb{R})$, establishing

$$\|f\|_{L^2} = \frac{1}{\sqrt{2\pi}}\|\widehat{f}\|_{L^2}.$$

- **Hausdorff–Young Inequality:** Bounds $\|\widehat{f}\|_{L^q}$ in terms of $\|f\|_{L^p}$.
- These results expand Fourier analysis from L^1 to the entire L^p scale.

Further Directions (Not Covered Today)

- **Plancherel Theorem:** Extension of inversion to $L^2(\mathbb{R})$, establishing

$$\|f\|_{L^2} = \frac{1}{\sqrt{2\pi}}\|\widehat{f}\|_{L^2}.$$

- **Hausdorff–Young Inequality:** Bounds $\|\widehat{f}\|_{L^q}$ in terms of $\|f\|_{L^p}$.
- These results expand Fourier analysis from L^1 to the entire L^p scale.

$\Longrightarrow$ The inversion theorem is just the *beginning* of the story.

Plancherel: statement (lite)

Theorem. $\mathcal{F}$ extends uniquely to a unitary map on $L^2(\mathbb{R})$:

$$\|f\|_{L^2} = \|\widehat{f}\|_{L^2}, \qquad \langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle.$$

Why it matters: energy conservation; clean inversion on L^2 without assuming $\widehat{f} \in L^1$.

Plancherel: proof sketch in 4 beats

1. Prove on Schwartz $\mathcal{S}$: $\|\widehat{f}\|_2 = \|f\|_2$ (Gaussian integral + Fubini).
2. Density: $\mathcal{S}$ is dense in L^2.
3. Extend by continuity: define $\widehat{\cdot}$ on L^2 as the L^2-limit.
4. Unitarity and inversion follow; Parseval $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$.

Plancherel: quick example

If $f(x) = \mathrm{e}^{-\pi x^2}$, then $\widehat{f}(\xi) = \mathrm{e}^{-\pi \xi^2}$ and

$$\int_{\mathbb{R}} |f|^2 = \int_{\mathbb{R}} \mathrm{e}^{-2\pi x^2}\, dx = \int_{\mathbb{R}} |\widehat{f}|^2 \quad \text{(Gaussian is self-dual).}$$

Takeaway: L^2 is the “natural home” of Fourier analysis.

Hausdorff–Young (one-slide version)

Theorem. If $f \in L^p(\mathbb{R})$ with $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\widehat{f} \in L^q(\mathbb{R})$ and

$$\|\widehat{f}\|_{L^q} \le \|f\|_{L^p}.$$

Idea. Interpolate between the endpoints: $L^1 \to L^\infty$ (trivial bound) and $L^2 \to L^2$ (Plancherel).
Use. Controls $\widehat{f}$ integrability; key in estimates PDE.

Applications montage (pick 1 if time)

- Heat equation: $\partial_t \widehat{u} = -4\pi^2 |\xi|^2 \, \widehat{u} \Rightarrow u = G_t * u_0$ (Gaussian kernel).
- Probability: characteristic functions $\varphi_X(\xi) = \mathbb{E}[\mathrm{e}^{2\pi i \xi X}]$; CLT via pointwise product limits.
- Signals: convolutional smoothing = low-pass multiplier $\widehat{g}$; de-noising; deconvolution caveats.